Criteria for the L^p-dissipativity of systems of second order differential equations

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Abstract. We give complete algebraic characterizations of the L^p-dissipativity of the Dirichlet problem for some systems of partial differential operators of the form $\partial_h(\mathscr{A}^{hk}(x)\partial_k)$, were $\mathscr{A}^{hk}(x)$ are $m\times m$ matrices. First, we determine the sharp angle of dissipativity for a general scalar operator with complex coefficients. Next we prove that the two-dimensional elasticity operator is L^p-dissipative if and only if

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2},$$

ν being the Poisson ratio. Finally we find a necessary and sufficient algebraic condition for the L^p-dissipativity of the operator $\partial_h(\mathscr{A}^h(x)\partial_h)$, where $\mathscr{A}^h(x)$ are $m\times m$ matrices with complex $L^1_{\rm loc}$ entries, and we describe the maximum angle of L^p-dissipativity for this operator.

1 Introduction

Let Ω be a domain of $\mathbb{R}^n$ and let A be the operator

$$A=\partial_h(\mathscr{A}^{hk}(x)\partial_k) \tag{1.1}$$

where $\partial_k=\partial/\partial x_k$ and $\mathscr{A}^{hk}(x)=\{a_{ij}^{hk}(x)\}$ are $m\times m$ matrices whose elements are complex locally integrable functions defined in Ω $(1\leqslant i,j\leqslant$

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$m,\ 1 \leqslant h,k \leqslant 2)$. Here and in the sequel we adopt the summation convention and we put $p \in (1,\infty)$, $p' = p/(p-1)$. By $C_0^1(\Omega)$ we denote the space of all the C^1 functions having compact support in Ω.

Let $\mathscr{L}$ be the sesquilinear form related to the operator A

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{A}^{hk}(x)\partial_k u, \partial_h v\rangle \, dx.$$

($\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{C}^m$) defined in $(C_0^1(\Omega))^m \times (C_0^1(\Omega))^m$. We consider A as an operator acting from $(C_0^1(\Omega))^m$ to $((C_0^1(\Omega))^*)^m$ through the relation

$$\mathscr{L}(u,v) = -\int_\Omega \langle Au, v\rangle \, dx$$

for any $u,v \in (C_0^1(\Omega))^m$. Here the integration is understood in the sense of distributions.

Following [4], we say that the form $\mathscr{L}$ is L^p-dissipative if

$$\mathscr{R}e\, \mathscr{L}(u, |u|^{p-2}u) \geqslant 0 \qquad \text{if } p \geqslant 2, \tag{1.2}$$

$$\mathscr{R}e\, \mathscr{L}(|u|^{p'-2}u, u) \geqslant 0 \qquad \text{if } 1 < p < 2 \tag{1.3}$$

for all $u \in (C_0^1(\Omega))^m$. Unless otherwise stated we assume that the functions are complex vector valued.

Saying the L^p-dissipativity of the operator A, we mean the L^p-dissipativity of the corresponding form $\mathscr{L}$, just to simplify the terminology.

The problem of the dissipativity of linear differential operators and the problem of the contractivity of semigroups generated by them attracted much attention (see, e.g., [21, 3, 6, 1, 28, 7, 14, 26, 8, 9, 18, 19, 17, 16, 5, 13, 27, 20, 24, 22]). A detailed account of the subject can be found in the book [25], which contains also an extensive bibliography.

The present paper is devoted to the L^p-dissipativity $(1 < p < \infty)$ for partial differential operators. It is well known that scalar second order elliptic operators with real coefficients may generate contractive semigroups in L^p (see [21]). The case $p = \infty$ was considered in [15], where necessary and sufficient conditions for the L^∞-contractivity for scalar second order strongly elliptic systems with smooth coefficients were given. Necessary and sufficient conditions for the L^∞-contractivity were later given in [2] under the assumption that the coefficients are measurable and bounded.

The Dirichlet problem for the scalar operator (1.1) $(m = 1)$ is considered in [4] under the assumption that the entries of $\mathscr{A}$ are complex measures and

$\mathscr{I}m\,\mathscr{A}$ is symmetric. It is proved that the condition

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \qquad \forall\, \xi \in \mathbb{R}^n \tag{1.4}$$

is necessary and sufficient for the L^p-dissipativity.

The main results of the present work are as follows. In Section 2 we use (1.4) to obtain the sharp angle of dissipativity of a scalar complex differential operator A. To be more precise, we prove in Theorem 1 that zA ($z \in \mathbb{C}$) is L^p-dissipative if and only if $\vartheta_- \leqslant \arg z \leqslant \vartheta_+$, where ϑ_- and ϑ_+ are explicitly given (see (2.8)). Previously this result was known for operators with real coefficients (see [23] and Remark 1 below). It is worthwhile to remark that we never require ellipticity and we may deal with degenerate matrices.

In Section 3, the two-dimensional elasticity system is considered:

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u.$$

After proving a lemma concerning the L^p-dissipativity for general systems, it is shown that E is L^p-dissipative if and only if

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\ .$$

In Section 4 we deal with the class of systems of partial differential equations of the form

$$Au = \partial_h(\mathscr{A}^h(x)\partial_h u)$$

where $\mathscr{A}^h$ are $m\times m$ matrices whose elements are $L^1_{\rm loc}$ functions. We remark that the elasticity system is not of this form.

We find that the operator A is L^p-dissipative if and only if

$$\begin{gathered}\mathscr{R}e\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{R}e\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2\\ -(1-2/p)\,\mathscr{R}e(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle \geqslant 0\end{gathered}$$

for almost every $x \in \Omega$ and for any $\lambda,\omega \in \mathbb{C}^m$, $|\omega|=1$, $h=1,\ldots,n$. We determine also the angle of dissipativity for such operators.

In the particular case of positive real symmetric matrices $\mathscr{A}^h$, we prove that A is L^p-dissipative if and only if

$$\left(\frac{1}{2}-\frac{1}{p}\right)^2 (\mu_1^h(x)+\mu_m^h(x))^2 \leqslant \mu_1^h(x)\mu_m^h(x)$$

almost everywhere, $h=1,\ldots,n$, where $\mu_1^h(x)$ and $\mu_m^h(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}^h(x)$ respectively.

The results obtained in Section 4 are new even for systems of ordinary differential equations.

2 The angle of dissipativity of Second Order Scalar Complex Differential Operators

In this section we consider the operator

$$A = \nabla^t(\mathscr{A}(x)\nabla) \tag{2.1}$$

where $\mathscr{A} = \{a_{ij}(x)\}$ $(i,j = 1,\dots,n)$ is a matrix with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$. In [4] it is proved that, if $\mathscr{I}m\,\mathscr{A}$ is symmetric, there is the L^p-dissipativity of the Dirichlet problem for the differential operator A if and only if

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \tag{2.2}$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$.

For the sake of completeness we give a proof of the following elementary lemma

Lemma 1 *Let P and Q two real measurable functions defined on a set $\Omega \subset \mathbb{R}^n$. Let us suppose that $P(x) \geqslant 0$ almost everywhere. The inequality*

$$P(x)\,\cos\vartheta - Q(x)\,\sin\vartheta \geqslant 0 \qquad (\vartheta \in [-\pi,\pi]) \tag{2.3}$$

holds for almost every $x \in \Omega$ if and only if

$$\operatorname{arccot}\,[\operatorname*{ess\,inf}_{x\in\Xi}\,(Q(x)/P(x))] - \pi \leqslant \vartheta \leqslant \operatorname{arccot}\,[\operatorname*{ess\,sup}_{x\in\Xi}\,(Q(x)/P(x))] \tag{2.4}$$

where $\Xi = \{x \in \Omega \mid P^2(x) + Q^2(x) > 0\}$ and we set

$$Q(x)/P(x) = \begin{cases} +\infty & \text{if } P(x) = 0,\ Q(x) > 0 \\ -\infty & \text{if } P(x) = 0,\ Q(x) < 0. \end{cases}$$

Here $0 < \operatorname{arccot} y < \pi$, $\operatorname{arccot}(+\infty) = 0$, $\operatorname{arccot}(-\infty) = \pi$ and

$$\operatorname*{ess\,inf}_{x\in\Xi}\,(Q(x)/P(x)) = +\infty, \quad \operatorname*{ess\,sup}_{x\in\Xi}\,(Q(x)/P(x)) = -\infty$$

if Ξ has zero measure.

Proof. If Ξ has positive measure and $P(x) > 0$, inequality (2.3) means

$$\cos\vartheta - (Q(x)/P(x))\sin\vartheta \geqslant 0$$

and this is true if and only if

$$\operatorname{arccot}(Q(x)/P(x)) - \pi \leqslant \vartheta \leqslant \operatorname{arccot}(Q(x)/P(x)). \tag{2.5}$$

If $x \in \Xi$ and $P(x) = 0$, (2.3) means

$$-\pi \leqslant \vartheta \leqslant 0, \text{ if } Q(x) > 0, \quad 0 \leqslant \vartheta \leqslant \pi, \text{ if } Q(x) < 0.$$

This shows that (2.3) is equivalent to (2.5) provided that $x \in \Xi$. On the other hand, if $x \notin \Xi$, $P(x) = Q(x) = 0$ almost everywhere and (2.3) is always satisfied. Therefore, if Ξ has positive measure, (2.3) and (2.4) are equivalent.

If Ξ has zero measure, the result is trivial. $\square$

The next Theorem provides a necessary and sufficient condition for the L^p-dissipativity of the Dirichlet problem for the differential operator zA, where $z \in \mathbb{C}$.

Theorem 1 *Let the matrix $\mathscr{A}$ be symmetric. Let us suppose that the operator A is L^p-dissipative. Set*

$$\Lambda_1 = \operatorname*{ess\,inf}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle}{\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle}, \quad \Lambda_2 = \operatorname*{ess\,sup}_{(x,\xi)\in\Xi} \frac{\langle \mathscr{I}m\, \mathscr{A}(x)\xi,\xi\rangle}{\langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle}$$

where

$$\Xi = \{(x,\xi) \in \Omega \times \mathbb{R}^n \mid \langle \mathscr{R}e\, \mathscr{A}(x)\xi,\xi\rangle > 0\}. \tag{2.6}$$

The operator zA is L^p-dissipative if and only if

$$\vartheta_- \leqslant \arg z \leqslant \vartheta_+\,, \tag{2.7}$$

where

$$\begin{gathered} \vartheta_- = \begin{cases} \operatorname{arccot}\left(\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{|p-2|}\frac{1}{2\sqrt{p-1}+|p-2|\Lambda_1}\right) - \pi & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_1) - \pi & \text{if } p = 2 \end{cases} \\ \vartheta_+ = \begin{cases} \operatorname{arccot}\left(-\frac{2\sqrt{p-1}}{|p-2|} + \frac{p^2}{|p-2|}\frac{1}{2\sqrt{p-1}-|p-2|\Lambda_2}\right) & \text{if } p \neq 2 \\ \operatorname{arccot}(\Lambda_2) & \text{if } p = 2. \end{cases} \end{gathered} \tag{2.8}$$

Proof. The matrix $\mathscr{A}$ being symmetric, $\mathscr{I}m(e^{i\vartheta}A)$ is symmetric and in view of (2.2), the operator $e^{i\vartheta}A$ (with $\vartheta \in [-\pi,\pi]$) is L^p-dissipative if and only if

$$\begin{gathered}|p-2|\,|\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \sin\vartheta + \langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle \cos\vartheta| \leqslant \\ 2\sqrt{p-1}(\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \cos\vartheta - \langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle \sin\vartheta)\end{gathered} \tag{2.9}$$

for almost every $x\in\Omega$ and for any $\xi\in\mathbb{R}^n$. Suppose $p\neq 2$. Setting

$$\begin{aligned} a(x,\xi) &= |p-2|\,\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle, \quad & b(x,\xi) &= |p-2|\,\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle, \\ c(x,\xi) &= 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle, \quad & d(x,\xi) &= 2\sqrt{p-1}\,\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle, \end{aligned}$$

the inequality in (2.9) can be written as the system

$$\begin{cases} (c(x,\xi)-b(x,\xi))\cos\vartheta - (a(x,\xi)+d(x,\xi))\sin\vartheta \geqslant 0, \\ (c(x,\xi)+b(x,\xi))\cos\vartheta + (a(x,\xi)-d(x,\xi))\sin\vartheta \geqslant 0. \end{cases} \tag{2.10}$$

Noting that $c(x,\xi)\pm b(x,\xi)\geqslant 0$ because of (2.2), the solutions of the inequalities in (2.10) are given by the ϑ's satisfying both of the following conditions (see Lemma 1)

$$\begin{cases} \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\xi)\in\Xi_1} \frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}\right) - \pi \leqslant \vartheta \leqslant \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\xi)\in\Xi_1} \frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}\right) \\ \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\xi)\in\Xi_2} \frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}\right) - \pi \leqslant \vartheta \leqslant \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\xi)\in\Xi_2} \frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}\right) \end{cases}, \tag{2.11}$$

where

$$\begin{aligned} \Xi_1 &= \{(x,\xi)\in\Omega\times\mathbb{R}^n \mid (a(x,\xi)+d(x,\xi))^2 + (c(x,\xi)-b(x,\xi))^2 > 0\}, \\ \Xi_2 &= \{(x,\xi)\in\Omega\times\mathbb{R}^n \mid (a(x,\xi)-d(x,\xi))^2 + (b(x,\xi)+c(x,\xi))^2 > 0\}. \end{aligned}$$

We have

$$\begin{gathered} a(x,\xi)\,d(x,\xi) = b(x,\xi)\,c(x,\xi), \\ a^2(x,\xi)+b^2(x,\xi)+c^2(x,\xi)+d^2(x,\xi) = p^2(\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle^2 + \langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle^2) \end{gathered}$$

and then, keeping in mind (2.2), we may write $\Xi_1=\Xi_2=\Xi$, where Ξ is given by (2.6).

Moreover

$$\frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)} \geqslant \frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}$$

and then ϑ satisfies all of the inequalities in (2.11) if and only if

$$\operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\xi)\in\Xi} \tfrac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)}\right) - \pi \leqslant \vartheta \leqslant \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\xi)\in\Xi} \tfrac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)}\right) \qquad (2.12)$$

A direct computation shows that

$$\begin{aligned}
\frac{d(x,\xi)-a(x,\xi)}{c(x,\xi)+b(x,\xi)} &= \frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}+|p-2|\Lambda(x,\xi)}, \\
\frac{a(x,\xi)+d(x,\xi)}{c(x,\xi)-b(x,\xi)} &= -\frac{2\sqrt{p-1}}{|p-2|} + \frac{p^2}{|p-2|}\,\frac{1}{2\sqrt{p-1}-|p-2|\Lambda(x,\xi)}
\end{aligned}$$

where

$$\Lambda(x,\xi) = \frac{\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle}{\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle}\,.$$

Hence condition (2.12) is satisfied if and only if (2.7) holds.

If $p=2$, (2.9) is simply

$$\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \cos\vartheta - \langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle \sin\vartheta \geqslant 0$$

and the result follows directly from Lemma 1. $\square$

Remark 1 If $\mathscr{A}$ is a real matrix, then $\Lambda_1 = \Lambda_2 = 0$ and the angle of dissipativity does not depend on the operator. In fact we have

$$\frac{2\sqrt{p-1}}{|p-2|} - \frac{p^2}{2\sqrt{p-1}|p-2|} = -\frac{|p-2|}{2\sqrt{p-1}}$$

and Theorem 1 shows that zA is dissipative if and only if

$$\operatorname{arccot}\left(-\frac{|p-2|}{2\sqrt{p-1}}\right) - \pi \leqslant \arg z \leqslant \operatorname{arccot}\left(\frac{|p-2|}{2\sqrt{p-1}}\right),$$

i.e.

$$|\arg z| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right).$$

This is a well known result (see, e.g., [10], [11], [23]).

3 Two-dimensional Elasticity

Let us consider the classical operator of two-dimensional elasticity

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla \operatorname{div} u \tag{3.1}$$

where ν is the Poisson ratio. It is well known that E is strongly elliptic if and only if either $\nu > 1$ or $\nu < 1/2$.

In this Section we give a necessary and sufficient condition for the L^p-dissipativity of operator (3.1).

We start giving a necessary condition for the L^p-dissipativity of the operator

$$A = \partial_h(\mathscr{A}^{hk}(x)\partial_k) \tag{3.2}$$

where $\mathscr{A}^{hk}(x) = \{a_{ij}^{hk}(x)\}$ are $m \times m$ matrices whose elements are complex locally integrable functions defined in an arbitrary domain Ω of $\mathbb{R}^2$ $(1 \leqslant i, j \leqslant m,\ 1 \leqslant h, k \leqslant 2)$.

The following lemma holds in any number of variables.

Lemma 2 *Let Ω be a domain of $\mathbb{R}^n$. The operator* (3.2) *is L^p-dissipative if and only if*

$$\begin{gathered}\int_\Omega \Big(\mathscr{R}e\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle \\ -(1-2/p)^2|v|^{-4}\,\mathscr{R}e\langle \mathscr{A}^{hk}\, v, v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle \\ -(1-2/p)|v|^{-2}\,\mathscr{R}e(\langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle \\ -\langle \mathscr{A}^{hk}\,\partial_k v, v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle)\Big)dx \geqslant 0\end{gathered} \tag{3.3}$$

for any $v \in (C_0^1(\Omega))^m$. Here and in the sequel the integrand is extended by zero on the set where v vanishes.

Proof. *Sufficiency.* First suppose $p \geqslant 2$. Let $u \in (C_0^1(\Omega))^m$ and set $v = |u|^{(p-2)/2}u$. We have $v \in (C_0^1(\Omega))^m$ and $u = |v|^{(2-p)/p}v$. From the identities

$$\begin{gathered}\langle \mathscr{A}^{hk}\,\partial_k u, \partial_h(|u|^{p-2}u)\rangle = \\ \langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle - (1-2/p)^2|v|^{-2}\,\mathscr{R}e\langle \mathscr{A}^{hk}\, v, v\rangle\, \partial_k|v|\partial_h|v| \\ -(1-2/p)|v|^{-1}\,\mathscr{R}e(\langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\, \partial_k|v| - \langle \mathscr{A}^{hk}\,\partial_k v, v\rangle \partial_h|v|),\end{gathered}$$

$$\partial_k|v| = |v|^{-1}\,\mathscr{R}e\langle v, \partial_k v\rangle,$$

we see that the left-hand side in (3.3) is equal to $\mathscr{L}(u, |u|^{p-2}u)$. Then (1.2) is satisfied for any $u \in (C_0^1(\Omega))^m$.

If $1 < p < 2$, we may write (1.3) as

$$\mathscr{R}e \int_\Omega \langle (\mathscr{A}^{hk})^* \partial_h u, \partial_k (|u|^{p'-2} u)\rangle \, dx \geqslant 0$$

for any $u \in (C_0^1(\Omega))^m$. The first part of the proof shows that this implies

$$\begin{gathered}\int_\Omega \Big(\mathscr{R}e\langle (\mathscr{A}^{hk})^* \partial_h v, \partial_k v\rangle \\ -(1-2/p')^2 |v|^{-4} \mathscr{R}e\langle (\mathscr{A}^{hk})^* v, v\rangle \, \mathscr{R}e\langle v, \partial_h v\rangle \, \mathscr{R}e\langle v, \partial_k v\rangle \\ -(1-2/p')|v|^{-2} \, \mathscr{R}e(\langle (\mathscr{A}^{hk})^* v, \partial_k v\rangle \, \mathscr{R}e\langle v, \partial_h v\rangle \\ -\langle (\mathscr{A}^{hk})^* \partial_h v, v\rangle \, \mathscr{R}e\langle v, \partial_k v\rangle)\Big) dx \geqslant 0\end{gathered} \tag{3.4}$$

for any $v \in (C_0^1(\Omega))^m$. Since $1 - 2/p' = -(1-2/p)$, this inequality is exactly (3.3).

Necessity. Let $p \geqslant 2$ and set

$$g_\varepsilon = (|v|^2 + \varepsilon^2)^{1/2}, \quad u_\varepsilon = g_\varepsilon^{2/p-1} v,$$

where $v \in (C_0^1(\Omega))^m$. We have

$$\begin{gathered}\langle \mathscr{A}^{hk} \, \partial_k u_\varepsilon, \partial_h (|u_\varepsilon|^{p-2} u_\varepsilon)\rangle \\ = |u_\varepsilon|^{p-2} \langle \mathscr{A}^{hk} \, \partial_k u_\varepsilon, \partial_h u_\varepsilon\rangle + (p-2)|u_\varepsilon|^{p-3} \langle \mathscr{A}^{hk} \, \partial_k u_\varepsilon, u_\varepsilon\rangle \partial_h |u_\varepsilon|.\end{gathered}$$

One checks directly that

$$\begin{gathered}|u_\varepsilon|^{p-2} \langle \mathscr{A}^{hk} \, \partial_k u_\varepsilon, \partial_h u_\varepsilon\rangle \\ = (1-2/p)^2 g_\varepsilon^{-(p+2)} |v|^{p-2} \langle \mathscr{A}^{hk} \, v, v\rangle \; \mathscr{R}e\langle v, \partial_k v\rangle \; \mathscr{R}e\langle v, \partial_h v\rangle \\ -(1-2/p) g_\varepsilon^{-p} |v|^{p-2} (\langle \mathscr{A}^{hk} \, v, \partial_h v\rangle \, \mathscr{R}e\langle v, \partial_k v\rangle + \langle \mathscr{A}^{hk} \, \partial_k v, v\rangle \, \mathscr{R}e\langle v, \partial_h v\rangle) \\ + g_\varepsilon^{2-p} |v|^{p-2} \langle \mathscr{A}^{hk} \, \partial_k v, \partial_h v\rangle, \\ |u_\varepsilon|^{p-3} \langle \mathscr{A}^{hk} \, \partial_k u_\varepsilon, u_\varepsilon\rangle \, \partial_h |u_\varepsilon| \\ = (1-2/p)[(1-2/p) g_\varepsilon^{-(p+2)} |v|^{p-2} \\ - g_\varepsilon^{-p} |v|^{p-4}] \, \langle \mathscr{A}^{hk} \, v, v\rangle \; \mathscr{R}e\langle v, \partial_k v\rangle \; \mathscr{R}e\langle v, \partial_h v\rangle \\ + [g_\varepsilon^{2-p} |v|^{p-4} - (1-2/p) g_\varepsilon^{-p} |v|^{p-2}] \, \langle \mathscr{A}^{hk} \, \partial_k v, v\rangle \; \mathscr{R}e\langle v, \partial_h v\rangle\end{gathered}$$

on the set $E = \{x \in \Omega \mid |v(x)| > 0\}$. The inequality $g_\varepsilon^a \leqslant |v|^a$ for $a \leqslant 0$, shows that the right-hand sides are majorized by L^1 functions. Since $g_\varepsilon \to |v|$

pointwise as $\varepsilon \to 0^+$, we find

$$
\begin{gathered}
\lim_{\varepsilon\to 0^+} \langle \mathscr{A}^{hk}\, \partial_k u_\varepsilon, \partial_h(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle \\
= \langle \mathscr{A}^{hk}\, \partial_k v, \partial_h v\rangle - (1-2/p)^2 |v|^{-4} \langle \mathscr{A}^{hk}\, v, v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle \\
-(1-2/p)|v|^{-2}(\langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle - \langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle)
\end{gathered}
$$

and dominated convergence gives

$$
\begin{gathered}
\lim_{\varepsilon\to 0^+} \mathscr{L}(u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon) = \lim_{\varepsilon\to 0^+} \int_E \langle \mathscr{A}^{hk}\, \partial_k u_\varepsilon, \partial_h(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle\, dx = \\
\mathscr{R}e \int_E [\langle \mathscr{A}^{hk}\, \partial_k v, \partial_h v\rangle \\
-(1-2/p)^2 |v|^{-4} \langle \mathscr{A}^{hk}\, v, v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle \\
-(1-2/p)|v|^{-2}(\langle \mathscr{A}^{hk}\, v, \partial_h v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle \\
-\langle \mathscr{A}^{hk}\, \partial_k v, v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle)]\, dx.
\end{gathered}
\tag{3.5}
$$

The function u_ε being in $(C^1_0(\Omega))^m$, (1.2) implies (3.3).

If $1 < p < 2$, from (3.5) it follows that

$$
\begin{gathered}
\lim_{\varepsilon\to 0^+} \mathscr{L}(|u_\varepsilon|^{p'-2}u_\varepsilon, u_\varepsilon) = \lim_{\varepsilon\to 0^+} \mathscr{R}e \int_E \langle (\mathscr{A}^{hk})^* \partial_h u, \partial_k(|u|^{p'-2}u)\rangle\, dx = \\
\int_E \Big(\mathscr{R}e\langle (\mathscr{A}^{hk})^* \partial_h v, \partial_k v\rangle \\
-(1-2/p')^2 |v|^{-4}\, \mathscr{R}e\langle (\mathscr{A}^{hk})^* v, v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle \\
-(1-2/p')|v|^{-2}\, \mathscr{R}e(\langle (\mathscr{A}^{hk})^* v, \partial_k v\rangle\, \mathscr{R}e\langle v, \partial_h v\rangle \\
-\langle (\mathscr{A}^{hk})^* \partial_h v, v\rangle\, \mathscr{R}e\langle v, \partial_k v\rangle)\Big)\, dx.
\end{gathered}
$$

This shows that (1.3) implies (3.4) and the proof is complete. $\square$

Theorem 2 *Let Ω be a domain of $\mathbb{R}^2$. If the operator (3.2) is L^p-dissipative, we have*

$$
\begin{gathered}
\mathscr{R}e\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2 \\
-(1-2/p)\, \mathscr{R}e(\langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\omega, \lambda\rangle - \langle (\mathscr{A}^{hk}(x)\xi_h\xi_k)\lambda, \omega\rangle)\, \mathscr{R}e\langle \lambda, \omega\rangle \\
\geqslant 0
\end{gathered}
\tag{3.6}
$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^2$, λ, $\omega \in \mathbb{C}^m$, $|\omega| = 1$.

Proof. Let us assume that $\mathscr{A}$ is a constant matrix and that $\Omega = \mathbb{R}^2$. Let us fix $\omega \in \mathbb{C}^m$ with $|\omega| = 1$ and take $v(x) = w(x)\,\eta(\log|x|/\log R)$, where

$$w(x) = \mu\omega + \psi(x), \tag{3.7}$$

$\mu,\ R \in \mathbb{R}^+,\ R > 1,\ \psi \in (C_0^\infty(\mathbb{R}^2))^m,\ \eta \in C^\infty(\mathbb{R}),\ \eta(t) = 1$ if $t \leqslant 1/2$ and $\eta(t) = 0$ if $t \geqslant 1$.

We have

$$\begin{gathered}\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle = \langle \mathscr{A}^{hk}\,\partial_k w, \partial_h w\rangle \eta^2(\log|x|/\log R)\\ +(\log R)^{-1}(\langle \mathscr{A}^{hk}\,\partial_k w, w\rangle x_h + \langle \mathscr{A}^{hk}\,w, \partial_h w\rangle x_k)\times\\ |x|^{-2}\eta(\log|x|/\log R)\,\eta'(\log|x|/\log R)\\ +(\log R)^{-2}\langle \mathscr{A}^{hk}\,w, w\rangle x_h x_k |x|^{-4}\,(\eta'(\log|x|/\log R))^2\end{gathered}$$

and then, choosing δ such that $\operatorname{spt}\psi \subset B_\delta(0)$,

$$\begin{gathered}\int_{\mathbb{R}^2}\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle dx = \int_{B_\delta(0)}\langle \mathscr{A}^{hk}\,\partial_k w, \partial_h w\rangle dx\\ +\frac{1}{\log^2 R}\int_{B_R(0)\setminus B_{\sqrt{R}}(0)}\langle \mathscr{A}^{hk}\,w, w\rangle \frac{x_h x_k}{|x|^4}\,(\eta'(\log|x|/\log R))^2\,dx\end{gathered}$$

provided that $R > \delta^2$. Since

$$\lim_{R\to+\infty}\frac{1}{\log^2 R}\int_{B_R(0)\setminus B_{\sqrt{R}}(0)}\frac{dx}{|x|^2} = 0,$$

we have

$$\lim_{R\to+\infty}\int_{\mathbb{R}^2}\langle \mathscr{A}^{hk}\,\partial_k v, \partial_h v\rangle dx = \int_{B_\delta(0)}\langle \mathscr{A}^{hk}\,\partial_k w, \partial_h w\rangle dx.$$

On the set where $v \neq 0$ we have

$$\begin{gathered}|v|^{-4}\langle \mathscr{A}^{hk}\,v, v\rangle\,\mathscr{R}e\langle v, \partial_k v\rangle\,\mathscr{R}e\langle v, \partial_h v\rangle =\\ |w|^{-4}\langle \mathscr{A}^{hk}\,w, w\rangle\,\mathscr{R}e\langle w, \partial_k w\rangle\,\mathscr{R}e\langle w, \partial_h w\rangle \eta^2(\log|x|/\log R)\\ +(\log R)^{-1}|w|^{-2}\langle \mathscr{A}^{hk}\,w, w\rangle(\mathscr{R}e\langle w, \partial_h w\rangle x_k + \mathscr{R}e\langle w, \partial_k w\rangle x_h)|x|^{-2}\times\\ \eta(\log|x|/\log R)\,\eta'(\log|x|/\log R)\\ +(\log R)^{-2}\langle \mathscr{A}^{hk}\,w, w\rangle x_h x_k |x|^{-4}(\eta'(\log|x|/\log R))^2\end{gathered}$$

and then

$$\lim_{R\to+\infty}\int_{\mathbb{R}^2}|v|^{-4}\langle\mathscr{A}^{hk}\,v,v\rangle\,\mathscr{R}e\langle v,\partial_k v\rangle\,\mathscr{R}e\langle v,\partial_h v\rangle dx =$$
$$\int_{B_\delta(0)}|w|^{-4}\langle\mathscr{A}^{hk}\,w,w\rangle\,\mathscr{R}e\langle w,\partial_k w\rangle\,\mathscr{R}e\langle w,\partial_h w\rangle dx.$$

In the same way we obtain

$$\lim_{R\to+\infty}\int_{\mathbb{R}^2}|v|^{-2}\,\mathscr{R}e(\langle\mathscr{A}^{hk}\,v,\partial_h v\rangle\,\mathscr{R}e\langle v,\partial_k v\rangle-\langle\mathscr{A}^{hk}\,\partial_k v,v\rangle\,\mathscr{R}e\langle v,\partial_h v\rangle)dx =$$
$$\int_{B_\delta(0)}|w|^{-2}\,\mathscr{R}e(\langle\mathscr{A}^{hk}\,w,\partial_h w\rangle\,\mathscr{R}e\langle w,\partial_k w\rangle-\langle\mathscr{A}^{hk}\,\partial_k w,w\rangle\,\mathscr{R}e\langle w,\partial_h w\rangle)dx.$$

In view of Lemma 2, (3.3) holds. Putting v in this formula and letting $R\to+\infty$, we find

$$\begin{gathered}\int_{B_\delta(0)}\Big(\mathscr{R}e\langle\mathscr{A}^{hk}\,\partial_k w,\partial_h w\rangle\\ -(1-2/p)^2|w|^{-4}\,\mathscr{R}e\langle\mathscr{A}^{hk}\,w,w\rangle\,\mathscr{R}e\langle w,\partial_k w\rangle\,\mathscr{R}e\langle w,\partial_h w\rangle\\ -(1-2/p)|w|^{-2}\,\mathscr{R}e(\langle\mathscr{A}^{hk}\,w,\partial_h w\rangle\,\mathscr{R}e\langle w,\partial_k w\rangle\\ -\langle\mathscr{A}^{hk}\,\partial_k w,w\rangle\,\mathscr{R}e\langle w,\partial_h w\rangle)\Big)dx\geqslant 0.\end{gathered}\tag{3.8}$$

On the other hand, keeping in mind (3.7),

$$\begin{gathered}\mathscr{R}e\langle\mathscr{A}^{hk}\,\partial_k w,\partial_h w\rangle=\mathscr{R}e\langle\mathscr{A}^{hk}\,\partial_k\psi,\partial_h\psi\rangle,\\ |w|^{-4}\,\mathscr{R}e\langle\mathscr{A}^{hk}\,w,w\rangle\,\mathscr{R}e\langle w,\partial_k w\rangle\,\mathscr{R}e\langle w,\partial_h w\rangle=\\ |\mu\omega+\psi|^{-4}\,\mathscr{R}e\langle\mathscr{A}^{hk}(\mu\omega+\psi),\mu\omega+\psi\rangle\,\mathscr{R}e\langle\mu\omega+\psi,\partial_k\psi\rangle\,\mathscr{R}e\langle\mu\omega+\psi,\partial_h\psi\rangle,\\ |w|^{-2}\,\mathscr{R}e(\langle\mathscr{A}^{hk}\,w,\partial_h w\rangle\,\mathscr{R}e\langle w,\partial_k w\rangle-\langle\mathscr{A}^{hk}\,\partial_k w,w\rangle\,\mathscr{R}e\langle w,\partial_h w\rangle)=\\ |\mu\omega+\psi|^{-2}\,\mathscr{R}e(\langle\mathscr{A}^{hk}(\mu\omega+\psi),\partial_h\psi\rangle\,\mathscr{R}e\langle\mu\omega+\psi,\partial_k\psi\rangle\\ -\langle\mathscr{A}^{hk}\,\partial_k(\mu\omega+\psi),\mu\omega+\psi\rangle\,\mathscr{R}e\langle\mu\omega+\psi,\partial_h\psi\rangle).\end{gathered}$$

Letting $\mu\to+\infty$ in (3.8), we obtain

$$\begin{gathered}\int_{\mathbb{R}^2}\Big(\mathscr{R}e\langle\mathscr{A}^{hk}\,\partial_k\psi,\partial_h\psi\rangle\\ -(1-2/p)^2\,\mathscr{R}e\langle\mathscr{A}^{hk}\,\omega,\omega\rangle\,\mathscr{R}e\langle\omega,\partial_k\psi\rangle\,\mathscr{R}e\langle\omega,\partial_h\psi\rangle\\ -(1-2/p)\,\mathscr{R}e(\langle\mathscr{A}^{hk}\,\omega,\partial_h\psi\rangle\,\mathscr{R}e\langle\omega,\partial_k\psi\rangle\\ -\langle\mathscr{A}^{hk}\,\partial_k\psi,\omega\rangle\,\mathscr{R}e\langle\omega,\partial_h\psi\rangle)\Big)\,dx\geqslant 0.\end{gathered}\tag{3.9}$$

Putting in (3.9)

$$\psi(x) = \lambda\, \varphi(x)\, e^{i\mu\langle\xi,x\rangle}$$

where $\lambda \in \mathbb{C}^m$, $\varphi \in C_0^\infty(\mathbb{R}^2)$ and μ is a real parameter, by standard arguments (see, e.g., [12, p.107–108]), we find (3.6).

If the matrix $\mathscr{A}$ is not constant, take $\psi \in (C_0^1(\mathbb{R}^2))^m$ and define

$$v(x) = \psi((x - x_0)/\varepsilon)$$

where x_0 is a fixed point in Ω and $0 < \varepsilon < \operatorname{dist}(x_0, \partial\Omega)$.

Putting this particular v in (3.3) and making a change of variables, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \Big(& \mathscr{R}e\langle \mathscr{A}^{hk}(x_0+\varepsilon y)\partial_k\psi, \partial_h\psi\rangle \\ & -(1-2/p)^2|\psi|^{-4}\, \mathscr{R}e\langle \mathscr{A}^{hk}(x_0+\varepsilon y)\psi,\psi\rangle\, \mathscr{R}e\langle\psi,\partial_k\psi\rangle\, \mathscr{R}e\langle\psi,\partial_h\psi\rangle \\ & -(1-2/p)|\psi|^{-2}\, \mathscr{R}e(\langle \mathscr{A}^{hk}(x_0+\varepsilon y)\psi,\partial_h\psi\rangle\, \mathscr{R}e\langle\psi,\partial_k\psi\rangle \\ & -\langle \mathscr{A}^{hk}(x_0+\varepsilon y)\partial_k\psi,\psi\rangle\, \mathscr{R}e\langle\psi,\partial_h\psi\rangle)\Big)\, dy \geqslant 0.\end{aligned}$$

Letting $\varepsilon \to 0^+$ we find

$$\begin{aligned}\int_{\mathbb{R}^2} \Big(& \mathscr{R}e\langle \mathscr{A}^{hk}(x_0)\partial_k\psi, \partial_h\psi\rangle \\ & -(1-2/p)^2|\psi|^{-4}\, \mathscr{R}e\langle \mathscr{A}^{hk}(x_0)\psi,\psi\rangle\, \mathscr{R}e\langle\psi,\partial_k\psi\rangle\, \mathscr{R}e\langle\psi,\partial_h\psi\rangle \\ & -(1-2/p)|\psi|^{-2}\, \mathscr{R}e(\langle \mathscr{A}^{hk}(x_0)\psi,\partial_h\psi\rangle\, \mathscr{R}e\langle\psi,\partial_k\psi\rangle \\ & -\langle \mathscr{A}^{hk}(x_0)\partial_k\psi,\psi\rangle\, \mathscr{R}e\langle\psi,\partial_h\psi\rangle)\Big)\, dy \geqslant 0\end{aligned}$$

for almost every $x_0 \in \Omega$. The arbitrariness of $\psi \in (C_0^1(\mathbb{R}^2))^m$ and what we have proved for constant matrices give the result. □

Since in problem of Elasticity we are interested in real solutions, we shall discuss the L^p-dissipativity of the operator (3.1) in a real frame. In the present Section, all the functions we are going to consider, in particular the ones appearing in the conditions (1.2) and (1.3), are supposed to be real vector valued.

Theorem 3 *The operator* (3.1) *is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}. \tag{3.10}$$

Proof. *Necessity.* We have

$$\langle(\mathscr{A}^{hk}\,\xi_h\xi_k)\lambda,\lambda\rangle = |\xi|^2|\lambda|^2 + (1-2\nu)^{-1}\langle\xi,\lambda\rangle^2,$$
$$\langle(\mathscr{A}^{hk}\,\xi_h\xi_k)\omega,\omega\rangle = |\xi|^2 + (1-2\nu)^{-1}\langle\xi,\omega\rangle^2,$$
$$\langle(\mathscr{A}^{hk}\,\xi_h\xi_k)\lambda,\omega\rangle = |\xi|^2|\langle\lambda,\omega\rangle + (1-2\nu)^{-1}\langle\xi,\lambda\rangle\langle\xi,\omega\rangle$$

for any ξ, λ, $\omega \in \mathbb{R}^2$, $|\omega| = 1$. Hence, in view of Theorem 2, the L^p-dissipativity of E implies

$$\begin{gathered}-(1-2/p)^2[|\xi|^2 + (1-2\nu)^{-1}(\xi_j\omega_j)^2](\lambda_j\omega_j)^2\\ +|\xi|^2|\lambda|^2 + (1-2\nu)^{-1}(\xi_j\lambda_j)^2 \geqslant 0\end{gathered} \tag{3.11}$$

for any ξ, λ, $\omega\in\mathbb{R}^2$, $|\omega|=1$.

Without loss of generality we may suppose $\xi = (1,0)$. Setting $C_p = (1-2/p)^2$ and $\gamma = (1-2\nu)^{-1}$, condition (3.11) can be written as

$$-\,C_p(1+\gamma\omega_1^2)(\lambda_j\omega_j)^2 + |\lambda|^2 + \gamma\lambda_1^2 \geqslant 0 \tag{3.12}$$

for any λ, $\omega\in\mathbb{R}^2$, $|\omega|=1$.

Condition (3.12) holds if and only if

$$\begin{gathered}-C_p(1+\gamma\omega_1^2)\omega_1^2 + 1 + \gamma \geqslant 0,\\ [C_p(1+\gamma\omega_1^2)\omega_1\omega_2]^2 \leqslant\\ [-C_p(1+\gamma\omega_1^2)\omega_1^2 + 1 + \gamma]\,[-C_p(1+\gamma\omega_1^2)\omega_2^2 + 1]\end{gathered}$$

for any $\omega\in\mathbb{R}^2$, $|\omega|=1$.

In particular, the second condition has to be satisfied. This can be written in the form

$$1+\gamma - C_p(1+\gamma\omega_1^2)(1+\gamma\omega_2^2) \geqslant 0 \tag{3.13}$$

for any $\omega\in\mathbb{R}^2$, $|\omega|=1$. The minimum of the left hand side of (3.13) on the unit sphere is given by

$$1+\gamma - C_p(1+\gamma/2)^2.$$

Hence (3.13) is satisfied if and only if $1+\gamma - C_p(1+\gamma/2)^2 \geqslant 0$. The last inequality means

$$\frac{2(1-\nu)}{1-2\nu} - \left(\frac{p-2}{p}\right)^2\left(\frac{3-4\nu}{2(1-2\nu)}\right)^2 \geqslant 0$$

i.e. (3.10). From the identity $4/(p\,p') = 1-(1-2/p)^2$, it follows that (3.10) can be written also as

$$\frac{4}{p\,p'} \geqslant \frac{1}{(3-4\nu)^2}. \tag{3.14}$$

Sufficiency. In view of Lemma 2, E is L^p-dissipative if and only if

$$\int_\Omega [-C_p|\nabla|v||^2 + \sum_j |\nabla v_j|^2 - \gamma\, C_p\, |v|^{-2}|v_h\partial_h|v||^2 + \gamma\,|\operatorname{div} v|^2]\, dx \geqslant 0 \tag{3.15}$$

for any $v \in (C_0^1(\Omega))^2$. Choose $v \in (C_0^1(\Omega))^2$ and define

$$X_1 = |v|^{-1}(v_1\partial_1|v| + v_2\partial_2|v|), \quad X_2 = |v|^{-1}(v_2\partial_1|v| - v_1\partial_2|v|)$$
$$Y_1 = |v|[\partial_1(|v|^{-1}v_1) + \partial_2(|v|^{-1}v_2)], \quad Y_2 = |v|[\partial_1(|v|^{-1}v_2) - \partial_2(|v|^{-1}v_1)]$$

on the set $E = \{x \in \Omega \mid v \neq 0\}$. From the identities

$$|\nabla|v||^2 = X_1^2 + X_2^2, \quad Y_1 = (\partial_1 v_1 + \partial_2 v_2) - X_1, \quad Y_2 = (\partial_1 v_2 - \partial_2 v_1) - X_2$$

it follows

$$\begin{aligned} Y_1^2 + Y_2^2 = |\nabla|v||^2 + (\partial_1 v_1 + \partial_2 v_2)^2 + (\partial_1 v_2 - \partial_2 v_1)^2 \\ -2(\partial_1 v_1 + \partial_2 v_2)X_1 - 2(\partial_1 v_2 - \partial_2 v_1)X_2. \end{aligned}$$

Keeping in mind that $\partial_h|v| = |v|^{-1}v_j\partial_h v_j$, one can check that

$$\begin{aligned} (\partial_1 v_1 + \partial_2 v_2)(v_1\partial_1|v| + v_2\partial_2|v|) + (\partial_1 v_2 - \partial_2 v_1)(v_2\partial_1|v| - v_1\partial_2|v|) = \\ |v|\,|\nabla|v||^2 + |v|(\partial_1 v_1\partial_2 v_2 - \partial_2 v_1\partial_1 v_2), \end{aligned}$$

which implies

$$\sum_j |\nabla v_j|^2 = X_1^2 + X_2^2 + Y_1^2 + Y_2^2. \tag{3.16}$$

Thus (3.15) can be written as

$$\int_E \left[\frac{4}{p\,p'}(X_1^2 + X_2^2) + Y_1^2 + Y_2^2 - \gamma\, C_p X_1^2 + \gamma\,(X_1 + Y_1)^2\right] dx \geqslant 0. \tag{3.17}$$

Let us prove that

$$\int_E X_1 Y_1 dx = -\int_E X_2 Y_2 dx. \tag{3.18}$$

Since $X_1 + Y_1 = \operatorname{div} v$ and $X_2 + Y_2 = \partial_1 v_2 - \partial_2 v_1$, keeping in mind (3.16), we may write

$$2\int_E (X_1Y_1+X_2Y_2)dx = \int_E [(X_1+Y_1)^2+(X_2+Y_2)^2-(X_1^2+X_2^2+Y_1^2+Y_2^2)]\,dx = \int_E [(\operatorname{div} v)^2 + (\partial_1 v_2 - \partial_2 v_1)^2 - \sum_j |\nabla v_j|^2]\,dx$$

i.e.

$$\int_E (X_1Y_1 + X_2Y_2)dx = \int_E (\partial_1 v_1 \partial_2 v_2 - \partial_1 v_2 \partial_2 v_1)\,dx.$$

The set $\{x \in \Omega \setminus E \mid \nabla v(x) \neq 0\}$ has zero measure and then

$$\int_E (X_1Y_1 + X_2Y_2)dx = \int_\Omega (\partial_1 v_1 \partial_2 v_2 - \partial_1 v_2 \partial_2 v_1)\,dx.$$

There exists a sequence $\{v^{(n)}\} \subset C_0^\infty(\Omega)$ such that $v^{(n)} \to v$, $\nabla v^{(n)} \to \nabla v$ uniformly in Ω and hence

$$\int_\Omega \partial_1 v_1 \partial_2 v_2 dx = \lim_{n\to\infty} \int_\Omega \partial_1 v_1^{(n)} \partial_2 v_2^{(n)} dx = \lim_{n\to\infty} \int_\Omega \partial_1 v_2^{(n)} \partial_2 v_1^{(n)} dx = \int_\Omega \partial_1 v_2 \partial_2 v_1 dx$$

and (3.18) is proved. In view of this, (3.17) can be written as

$$\int_E \left(\frac{4}{p\,p'}(1+\gamma)X_1^2 + 2\vartheta\gamma\, X_1Y_1 + (1+\gamma)Y_1^2 \right) dx + \int_E \left(\frac{4}{p\,p'}X_2^2 - 2(1-\vartheta)\gamma\, X_2Y_2 + Y_2^2 \right) dx \geqslant 0$$

for any fixed $\vartheta \in \mathbb{R}$.

If we choose

$$\vartheta = \frac{2(1-\nu)}{3-4\nu}$$

we find

$$(1-\vartheta)\gamma = \frac{1}{3-4\nu}\,, \qquad \vartheta^2\gamma^2 = \frac{(1+\gamma)^2}{(3-4\nu)^2}\,.$$

Inequality (3.14) leads to

$$\vartheta^2\gamma^2 \leqslant \frac{4}{p\,p'}\,(1+\gamma)^2, \qquad (1-\vartheta)^2\gamma^2 \leqslant \frac{4}{p\,p'}\,.$$

Observing that (3.10) implies $1+\gamma = 2(1-\nu)(1-2\nu)^{-1} \geqslant 0$, we get

$$\frac{4}{p\,p'}(1+\gamma)x_1^2 + 2\vartheta\gamma\, x_1 y_1 + (1+\gamma)y_1^2 \geqslant 0,$$
$$\frac{4}{p\,p'}x_2^2 - 2(1-\vartheta)\gamma\, x_2 y_2 + y_2^2 \geqslant 0$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$. This shows that (3.17) holds. Then (3.15) is true for any $v \in (C_0^1(\Omega))^2$ and the proof is complete. □

We shall now give two Corollaries of this result. They concerns the comparison between E and Δ from the point of view of the L^p-dissipativity.

Corollary 1 *There exists $k > 0$ such that $E - k\Delta$ is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 < \frac{2(\nu-1)(2\nu-1)}{(3-4\nu)^2}\,. \tag{3.19}$$

Proof. *Necessity.* We remark that if $E - k\Delta$ is L^p-dissipative, then

$$\begin{cases} k \leqslant 1 & \text{if } p = 2 \\ k < 1 & \text{if } p \neq 2. \end{cases} \tag{3.20}$$

In fact, in view of Theorem 2, we have the necessary condition

$$\begin{gathered} -(1-2/p)^2[(1-k)|\xi|^2 + (1-2\nu)^{-1}(\xi_j\omega_j)^2](\lambda_j\omega_j)^2 \\ +(1-k)|\xi|^2|\lambda|^2 + (1-2\nu)^{-1}(\xi_j\lambda_j)^2 \geqslant 0 \end{gathered} \tag{3.21}$$

for any ξ, λ, $\omega \in \mathbb{R}^2$, $|\omega| = 1$. If we take $\xi = (1,0)$, $\lambda = \omega = (0,1)$ in (3.21) we find

$$\frac{4}{p\,p'}\,(1-k) \geqslant 0$$

and then $k \leqslant 1$ for any p. If $p \neq 2$ and $k = 1$, taking $\xi = (1,0)$, $\lambda = (0,1)$, $\omega = (1/\sqrt{2}, 1/\sqrt{2})$ in (3.21), we find $-(1-2/p)^2(1-2\nu)^{-1} \geqslant 0$. On the other hand, taking $\xi = \lambda = (1,0)$, $\omega = (0,1)$ we find $(1-2\nu)^{-1} \geqslant 0$. This is a contradiction and (3.20) is proved.

It is clear that if $E - k\Delta$ is L^p-dissipative, then $E - k'\Delta$ is L^p-dissipative for any $k' < k$. Therefore it is not restrictive to suppose that $E - k\Delta$ is L^p-dissipative for some $0 < k < 1$. Moreover E is also L^p-dissipative.

The L^p-dissipativity of $E - k\Delta$ $(0 < k < 1)$ is equivalent to the L^p-dissipativity of the operator

$$E'u = \Delta u + (1-k)^{-1}(1-2\nu)^{-1}\nabla \operatorname{div} u. \tag{3.22}$$

Setting

$$\nu' = \nu(1-k) + k/2, \tag{3.23}$$

we have $(1-k)(1-2\nu) = 1 - 2\nu'$. Theorem 3 shows that

$$\frac{4}{p\,p'} \geqslant \frac{1}{(3-4\nu')^2}. \tag{3.24}$$

Since $3 - 4\nu' = 3 - 4\nu - 2k(1-2\nu)$, condition (3.24) means $|3 - 4\nu - 2k(1-2\nu)| \geqslant \sqrt{p\,p'}/2$, i.e.

$$\left| k - \frac{3-4\nu}{2(1-2\nu)} \right| \geqslant \frac{\sqrt{p\,p'}}{4|1-2\nu|} \tag{3.25}$$

Note that the L^p-dissipativity of E implies that (3.10) holds. In particular we have $(3-4\nu)/(1-2\nu) > 0$. Hence (3.25) is satisfied if either

$$k \leqslant \frac{1}{2|1-2\nu|} \left(|3-4\nu| - \frac{\sqrt{p\,p'}}{2} \right) \tag{3.26}$$

or

$$k \geqslant \frac{1}{2|1-2\nu|} \left(|3-4\nu| + \frac{\sqrt{p\,p'}}{2} \right) \tag{3.27}$$

Since

$$\frac{|3-4\nu|}{2|1-2\nu|} - 1 = \frac{3-4\nu}{2(1-2\nu)} - 1 = \frac{1}{2(1-2\nu)} \geqslant -\frac{\sqrt{p\,p'}}{4|1-2\nu|}$$

we have

$$\frac{1}{2|1-2\nu|} \left(|3-4\nu| + \frac{\sqrt{p\,p'}}{2} \right) \geqslant 1$$

and (3.27) is impossible. Then (3.26) holds. Since $k > 0$, we have the strict inequality in (3.14) and (3.19) is proved.

Sufficiency. Suppose (3.19). Since

$$\frac{4}{p\,p'} > \frac{1}{(3-4\nu)^2}\,,$$

we can take k such that

$$0 < k < \frac{1}{2|1-2\nu|}\left(|3-4\nu| - \frac{\sqrt{p\,p'}}{2}\right). \tag{3.28}$$

Note that

$$\frac{|3-4\nu|}{2|1-2\nu|} - 1 = \frac{3-4\nu}{2(1-2\nu)} - 1 = \frac{1}{2(1-2\nu)} \leqslant \frac{\sqrt{p\,p'}}{4|1-2\nu|}.$$

This means

$$\frac{1}{2|1-2\nu|}\left(|3-4\nu| - \frac{\sqrt{p\,p'}}{2}\right) \leqslant 1$$

and then $k < 1$. Let ν' be given by (3.23). The L^p-dissipativity of $E - k\Delta$ is equivalent to the L^p-dissipativity of the operator E' defined by (3.22).

Condition (3.25) (i.e. (3.24)) follows from (3.28) and Theorem 3 gives the result. $\square$

Corollary 2 *There exists $k < 2$ such that $k\Delta - E$ is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 < \frac{2\nu(2\nu-1)}{(1-4\nu)^2}\,. \tag{3.29}$$

Proof. We may write $k\Delta - E = \widetilde{E} - \widetilde{k}\Delta$, where $\widetilde{k} = 2 - k$, $\widetilde{E} = \Delta + (1-2\widetilde{\nu})^{-1}\nabla\,\mathrm{div}$, $\widetilde{\nu} = 1 - \nu$. Theorem 1 shows that $\widetilde{E} - \widetilde{k}\Delta$ is L^p-dissipative if and only if

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 < \frac{2(\widetilde{\nu}-1)(2\widetilde{\nu}-1)}{(3-4\widetilde{\nu})^2}\,. \tag{3.30}$$

Condition (3.30) coincides with(3.29) and the the Corollary is proved. $\square$

4 Dissipativity for a class of Systems of Partial Differential Equations

In this Section we consider a particular class of operators (1.1), namely the operators

$$Au = \partial_h(\mathscr{A}^h(x)\partial_h u) \tag{4.1}$$

where $\mathscr{A}^h(x) = \{a^h_{ij}(x)\}$ $(i,j = 1,\dots,m)$ are matrices with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$ $(h = 1,\dots,n)$.

Our goal is to prove that A is L^p-dissipative if and only if the algebraic condition

$$\begin{gathered}\mathscr{R}e\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathscr{R}e\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2\\ -(1-2/p)\,\mathscr{R}e(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle \geqslant 0\end{gathered}$$

is satisfied for almost every $x \in \Omega$ and for every $\lambda,\omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1,\dots,n$. In order to obtain such a result, in the next subsections we study the dissipativity for some systems of ordinary differential equations.

4.1 Dissipativity for Systems of Ordinary Differential Equations

In this subsection we consider the operator A defined by

$$Au = (\mathscr{A}(x)u')' \tag{4.2}$$

where $\mathscr{A}(x) = \{a_{ij}(x)\}$ $(i,j = 1,\dots,m)$ is a matrix with complex locally integrable entries defined in the bounded or unbounded interval (a,b).

In this case the sesquilinear form $\mathscr{L}(u,v)$ is given by

$$\mathscr{L}(u,v) = \int_a^b \langle \mathscr{A}\, u', v'\rangle\, dx.$$

Lemma 3 *The operator A is L^p-dissipative if and only if*

$$\begin{gathered}\int_a^b \Big(\mathscr{R}e\langle \mathscr{A}\, v',v'\rangle - (1-2/p)^2|v|^{-4}\,\mathscr{R}e\langle \mathscr{A}\, v,v\rangle(\mathscr{R}e\langle v,v'\rangle)^2\\ -(1-2/p)|v|^{-2}\,\mathscr{R}e(\langle \mathscr{A}\, v,v'\rangle - \langle \mathscr{A}\, v',v\rangle)\,\mathscr{R}e\langle v,v'\rangle\Big)\,dx \geqslant 0\end{gathered} \tag{4.3}$$

for any $v \in (C^1_0((a,b)))^m$.

Proof. It is a particular case of Lemma 2. $\square$

Theorem 4 *The operator A is L^p-dissipative if and only if*

$$\begin{gathered}\mathscr{R}e\langle\mathscr{A}(x)\lambda,\lambda\rangle-(1-2/p)^2\,\mathscr{R}e\langle\mathscr{A}(x)\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2\\ -(1-2/p)\,\mathscr{R}e(\langle\mathscr{A}(x)\omega,\lambda\rangle-\langle\mathscr{A}(x)\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle\geqslant 0\end{gathered}\tag{4.4}$$

for almost every $x\in(a,b)$ and for any $\lambda,\omega\in\mathbb{C}^m$, $|\omega|=1$.

Proof. *Necessity.* First we prove the result assuming that the coefficients a_{ij} are constant and that $(a,b)=\mathbb{R}$.

Let us fix λ and ω in $\mathbb{C}^m$, with $|\omega|=1$, and choose $v(x)=\eta(x/R)\,w(x)$ where

$$w_j(x)=\begin{cases}\mu\omega_j & \text{if } x<0\\ \mu\omega_j+x^2(3-2x)\lambda_j & \text{if } 0\leqslant x\leqslant 1\\ \mu\omega_j+\lambda_j & \text{if } x>1,\end{cases}$$

$\mu,R\in\mathbb{R}^+$, $\eta\in C_0^\infty(\mathbb{R})$, $\operatorname{spt}\eta\subset[-1,1]$ and $\eta(x)=1$ if $|x|\leqslant 1/2$.

We have

$$\begin{gathered}\langle\mathscr{A}\,v',v'\rangle=\\ \langle\mathscr{A}\,w',w'\rangle(\eta(x/R))^2+R^{-1}(\langle\mathscr{A}\,w',w\rangle+\langle\mathscr{A}\,w,w'\rangle)\eta(x/R)\eta'(x/R)\\ +R^{-2}\langle\mathscr{A}\,w,w\rangle(\eta'(x/R))^2\end{gathered}$$

and then

$$\int_{\mathbb{R}}\langle\mathscr{A}\,v',v'\rangle\,dx=\int_0^1\langle\mathscr{A}\,w',w'\rangle\,dx+\frac{1}{R^2}\int_{-R}^{R}\langle\mathscr{A}\,w,w\rangle(\eta'(x/R))^2dx$$

provided that $R>2$. Since $\langle\mathscr{A}\,w,w\rangle$ is bounded, we have

$$\lim_{R\to+\infty}\int_{\mathbb{R}}\langle\mathscr{A}\,v',v'\rangle\,dx=\int_0^1\langle\mathscr{A}\,w',w'\rangle\,dx.$$

On the set where $v\neq 0$ we have

$$\begin{gathered}|v|^{-4}\langle\mathscr{A}\,v,v\rangle(\mathscr{R}e\langle v,v'\rangle)^2=|w|^{-4}\langle\mathscr{A}\,w,w\rangle(\mathscr{R}e\langle w,w'\rangle)^2(\eta(x/R))^2\\ +2\,R^{-1}|w|^{-2}\langle\mathscr{A}\,w,w\rangle\,\mathscr{R}e\langle w,w'\rangle\eta(x/R))\,\eta'(x/R)+R^{-2}\langle\mathscr{A}\,w,w\rangle(\eta'(x/R))^2\end{gathered}$$

form which it follows

$$\lim_{R\to+\infty}\int_{\mathbb{R}}|v|^{-4}\langle \mathscr{A}\, v,v\rangle(\mathscr{R}e\langle v,v'\rangle)^2 dx=\int_0^1 |w|^{-4}\langle \mathscr{A}\, w,w\rangle(\mathscr{R}e\langle w,w'\rangle)^2 dx.$$

In the same way we obtain

$$\lim_{R\to+\infty}\int_{\mathbb{R}}|v|^{-2}(\langle \mathscr{A}\, v,v'\rangle-\langle \mathscr{A}\, v',v\rangle)\,\mathscr{R}e\langle v,v'\rangle\, dx=$$
$$\int_0^1 |w|^{-2}(\langle \mathscr{A}\, w,w'\rangle-\langle \mathscr{A}\, w',w\rangle)\,\mathscr{R}e\langle w,w'\rangle\, dx.$$

Since $v\in(C_0^1(\mathbb{R}))^m$, we can put v in (4.3). Letting $R\to+\infty$, we find

$$\begin{gathered}\int_0^1 \Big(\mathscr{R}e\langle \mathscr{A}\, w',w'\rangle-(1-2/p)^2|w|^{-4}\,\mathscr{R}e\langle \mathscr{A}\, w,w\rangle(\mathscr{R}e\langle w,w'\rangle)^2\\ -(1-2/p)|w|^{-2}\,\mathscr{R}e(\langle \mathscr{A}\, w,w'\rangle-\langle \mathscr{A}\, w',w\rangle)\,\mathscr{R}e\langle w,w'\rangle\Big)\, dx\geqslant 0.\end{gathered}\tag{4.5}$$

On the interval $(0,1)$ we have

$$\begin{gathered}\langle \mathscr{A}\, w',w'\rangle=\langle \mathscr{A}\,\lambda,\lambda\rangle\, 36x^2(1-x)^2,\\ |w|^{-4}\langle \mathscr{A}\, w,w\rangle(\mathscr{R}e\langle w,w'\rangle)^2=|\mu\omega+x^2(3-2x)\lambda|^{-4}\times\\ (\mu^2\langle \mathscr{A}\,\omega,\omega\rangle+\mu(\langle \mathscr{A}\,\omega,\lambda\rangle+\langle \mathscr{A}\,\lambda,\omega\rangle)x^2(3-2x)+\langle \mathscr{A}\,\lambda,\lambda\rangle x^4(3-2x)^2)\times\\ [\mathscr{R}e(\mu\langle\omega,\lambda\rangle\, 6x(1-x)+|\lambda|^2 6x^3(3-2x)(1-x))]^2,\\ |w|^{-2}(\langle \mathscr{A}\, w,w'\rangle-\langle \mathscr{A}\, w',w\rangle)\,\mathscr{R}e\langle w,w'\rangle=|\mu\omega+x^2(3-2x)\lambda|^{-2}\times\\ \mu(\langle \mathscr{A}\,\omega,\lambda\rangle-\langle \mathscr{A}\,\lambda,\omega\rangle)\, 6x(1-x)\,\mathscr{R}e(\mu\langle\omega,\lambda\rangle\, 6x(1-x)+|\lambda|^2 6x^3(3-2x)(1-x)).\end{gathered}$$

Letting $\mu\to\infty$ in (4.5) we find

$$\begin{gathered}36\int_0^1 \Big(\mathscr{R}e\langle \mathscr{A}\,\lambda,\lambda\rangle-(1-2/p)^2\,\mathscr{R}e\langle \mathscr{A}\,\omega,\omega\rangle(\mathscr{R}e\langle\omega,\lambda\rangle)^2\\ -(1-2/p)\,\mathscr{R}e(\langle \mathscr{A}\,\omega,\lambda\rangle-\langle \mathscr{A}\,\lambda,\omega\rangle)\,\mathscr{R}e\langle\omega,\lambda\rangle\Big)\, x^2(1-x)^2 dx\geqslant 0\end{gathered}$$

and (4.4) is proved.

If a_{hk} are not necessarily constant, consider

$$v(x)=\varepsilon^{-1/2}\psi((x-x_0)/\varepsilon)$$

where x_0 is a fixed point in (a,b), $\psi\in(C_0^1(\mathbb{R}))^m$ and ε is sufficiently small.

In this case (4.3) shows that

$$\int_{\mathbb{R}} \Big(\mathscr{R}e\langle \mathscr{A}(x_0+\varepsilon y)\psi', \psi'\rangle - (1-2/p)^2|\psi|^{-4}\, \mathscr{R}e\langle \mathscr{A}(x_0+\varepsilon y)\psi, \psi\rangle (\mathscr{R}e\langle \psi, \psi'\rangle)^2$$
$$-(1-2/p)|\psi|^{-2}\, \mathscr{R}e(\langle \mathscr{A}(x_0+\varepsilon y)\psi, \psi'\rangle - \langle \mathscr{A}(x_0+\varepsilon y)\psi', \psi\rangle)\, \mathscr{R}e\langle \psi, \psi'\rangle \Big)\, dy \geqslant 0.$$

Letting $\varepsilon \to 0^+$ we find for almost every x_0

$$\int_{\mathbb{R}} \Big(\mathscr{R}e\langle \mathscr{A}(x_0)\psi', \psi'\rangle - (1-2/p)^2|\psi|^{-4}\, \mathscr{R}e\langle \mathscr{A}(x_0)\psi, \psi\rangle (\mathscr{R}e\langle \psi, \psi'\rangle)^2$$
$$-(1-2/p)|\psi|^{-2}\, \mathscr{R}e(\langle \mathscr{A}(x_0)\psi, \psi'\rangle - \langle \mathscr{A}(x_0)\psi', \psi\rangle)\, \mathscr{R}e\langle \psi, \psi'\rangle \Big)\, dy \geqslant 0.$$

Because this inequality holds for any $\psi \in C_0^1(\mathbb{R})$, what we have obtained for constant coefficients gives the result.

Sufficiency. It is clear that, if (4.4) holds, then the integrand in (4.3) is nonnegative almost everywhere and Lemma 3 gives the result. □

Corollary 3 *If the operator A is L^p-dissipative, then*

$$\mathscr{R}e\langle \mathscr{A}(x)\lambda, \lambda\rangle \geqslant 0$$

for almost every $x \in (a,b)$ and for any $\lambda \in \mathbb{C}^m$.

Proof. Fix $x \in (a,b)$ such that (4.4) holds for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$. For any $\lambda \in \mathbb{C}^m$, choose ω such that $\langle \lambda, \omega\rangle = 0$, $|\omega| = 1$. The result follows by putting ω in (4.4). □

It is interesting to compare the operator A with the operator $I(d^2/dx^2)$.

Corollary 4 *There exists $k > 0$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m \\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > 0 \tag{4.6}$$

where

$$P(x,\lambda,\omega) = \mathscr{R}e\langle \mathscr{A}(x)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle \mathscr{A}(x)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2$$
$$-(1-2/p)\, \mathscr{R}e(\langle \mathscr{A}(x)\omega, \lambda\rangle - \langle \mathscr{A}(x)\lambda, \omega\rangle)\, \mathscr{R}e\langle \lambda, \omega\rangle.$$

There exists $k > 0$ such that $kI(d^2/dx^2) - A$ is L^p-dissipative if and only if

$$\operatorname*{ess\,sup}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) < \infty. \tag{4.7}$$

Proof. In view of Theorem 4, $A - kI(d^2/dx^2)$ is L^p-dissipative if and only if

$$P(x,\lambda,\omega) - k|\lambda|^2 + k(1-2/p)^2(\mathscr{R}e\langle\lambda,\omega\rangle)^2 \geqslant 0$$

for almost every $x \in (a,b)$ and for any $\lambda,\omega \in \mathbb{C}^m$, $|\omega| = 1$. Since

$$|\lambda|^2 - (1-2/p)^2(\mathscr{R}e\langle\lambda,\omega\rangle)^2 \geqslant \frac{4}{p\,p'}\,|\lambda|^2, \tag{4.8}$$

we can find a positive k such that this is true if and only if

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\omega|=1}} \frac{P(x,\lambda,\omega)}{|\lambda|^2 - (1-2/p)^2(\mathscr{R}e\langle\lambda,\omega\rangle)^2} > 0. \tag{4.9}$$

On the other hand, inequality (4.8) shows that

$$\frac{P(x,\lambda,\omega)}{|\lambda|^2} \leqslant \frac{P(x,\lambda,\omega)}{|\lambda|^2 - (1-2/p)^2(\mathscr{R}e\langle\lambda,\omega\rangle)^2} \leqslant \frac{p\,p'}{4}\,\frac{P(x,\lambda,\omega)}{|\lambda|^2} \tag{4.10}$$

and then (4.9) and (4.6) are equivalent.

In the same way the operator $kI(d^2/dx^2) - A$ is L^p-dissipative if and only if

$$-P(x,\lambda,\omega) + k|\lambda|^2 - k(1-2/p)^2(\mathscr{R}e\langle\lambda,\omega\rangle)^2 \geqslant 0$$

for almost every $x \in (a,b)$ and for any $\lambda,\omega \in \mathbb{C}^m$, $|\omega| = 1$. We can find a positive k such that this is true if and only if

$$\operatorname*{ess\,sup}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\omega|=1}} \frac{P(x,\lambda,\omega)}{|\lambda|^2 - (1-2/p)^2(\mathscr{R}e\langle\lambda,\omega\rangle)^2} < \infty.$$

This inequality is equivalent to (4.7) because of (4.10). $\square$

Corollary 5 *There exists $k \in \mathbb{R}$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,inf}_{\substack{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\\ |\lambda|=|\omega|=1}} P(x,\lambda,\omega) > -\infty.$$

Proof. The result can be proved as in Corollary 4. $\square$

4.2 Real coefficient operators

In the following we need the lemma

Lemma 4 *Let* $0 < \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_m$. *We have*

$$\max_{\substack{\omega\in\mathbb{R}^m\\|\omega|=1}} [(\mu_h\omega_h^2)(\mu_k^{-1}\omega_k^2)] = \frac{(\mu_1+\mu_m)^2}{4\,\mu_1\mu_m}. \tag{4.11}$$

Proof. First we proof by induction on m that

$$\max_{\substack{\omega\in\mathbb{R}^m\\|\omega|=1}} [(\mu_h\omega_h^2)(\mu_k^{-1}\omega_k^2)] = \max_{1\leqslant i<j\leqslant m} \frac{(\mu_i+\mu_j)^2}{4\,\mu_i\mu_j}. \tag{4.12}$$

In the case $m=2$, (4.12) is equivalent to

$$\max_{\varphi\in[0,2\pi]}[\cos^4\varphi+\sin^4\varphi+(\mu_1\mu_2^{-1}+\mu_2\mu_1^{-1})\cos^2\varphi\,\sin^2\varphi] = \frac{(\mu_1+\mu_2)^2}{4\,\mu_1\mu_2},$$

which can be easily proved.

Let $m>2$ and suppose $\mu_1<\mu_2<\ldots<\mu_m$; the maximum of the left hand side of (4.12) is the maximum of the function

$$\mu_h\mu_k^{-1}x_hx_k$$

subject to the constraint $x\in K$, where $K=\{x\in\mathbb{R}^m \mid x_1+\ldots+x_m = 1,\ 0\leqslant x_j\leqslant 1\ (j=1,\ldots,m)\}$. To find the constrained maximum, we first examine the system

$$\begin{cases} \gamma_{hk}x_k-\lambda=0 & h=1,\ldots,m\\ x_1+\ldots+x_m=1 \end{cases} \tag{4.13}$$

with $0\leqslant x_j\leqslant 1$ $(j=1,\ldots,m)$, where λ is the Lagrange multiplier and $\gamma_{hk}=\mu_h\mu_k^{-1}+\mu_k\mu_h^{-1}$.

Consider the homogeneous system

$$\gamma_{hk}x_k=0 \quad (h=1,\ldots,m). \tag{4.14}$$

One checks directly that the vectors $x^{(k)}=(x_1^{(k)},\ldots,x_m^{(k)})$,

$$x_1^{(k)}=\frac{\mu_1}{\mu_k}\,\frac{\mu_k^2-\mu_2^2}{\mu_2^2-\mu_1^2},\quad x_2^{(k)}=\frac{\mu_2}{\mu_k}\,\frac{\mu_1^2-\mu_k^2}{\mu_2^2-\mu_1^2},\quad x_j^{(k)}=\delta_{jk}\ (j=3,\ldots,m)$$

for $k = 3, \dots, m$, are $m-2$ linearly independent eigensolutions of the system (4.14). On the other hand, the determinant

$$\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{vmatrix} = 4 - \gamma_{12}^2 = -\frac{(\mu_1^2 - \mu_2^2)^2}{\mu_1^2 \mu_2^2} < 0$$

and then the rank of the matrix $\{\gamma_{hk}\}$ is 2.

Therefore there exists a solution of the system

$$\gamma_{hk} x_k = \lambda \qquad (h = 1, \dots, m) \tag{4.15}$$

if and only if the vector $(\lambda, \dots, \lambda)$ is orthogonal to any eigensolution of the adjoint homogeneous system. Since the matrix $\{\gamma_{hk}\}$ is symmetric, there exists a solution of the system (4.15) if and only if

$$\lambda(x_1^{(k)} + \cdots + x_m^{(k)}) = 0 \tag{4.16}$$

for $k = 3, \dots, m$.

But

$$x_1^{(k)} + \cdots + x_m^{(k)} = -\frac{\mu_1 \mu_2 + \mu_k^2}{\mu_k(\mu_1 + \mu_2)} + 1 = -\frac{(\mu_k - \mu_1)(\mu_k - \mu_2)}{\mu_k(\mu_1 + \mu_2)} < 0$$

and (4.16) are satisfied if and only if $\lambda = 0$. This means that the system (4.15) is solvable only when $\lambda = 0$ and the solutions are given by

$$x = \sum_{k=3}^{m} u_k x^{(k)}$$

for arbitrary $u_k \in \mathbb{R}$. On the other hand we are looking for solutions of (4.13) with $0 \leqslant x_j \leqslant 1$. Since $x_j = u_j$ for $j = 3, \dots, m$, we have $u_j \geqslant 0$. This implies that

$$x_2 = \sum_{k=3}^{m} \frac{\mu_2}{\mu_k} \frac{\mu_1^2 - \mu_k^2}{\mu_2^2 - \mu_1^2} u_k \leqslant 0$$

and since we require $x_2 \geqslant 0$, we have $u_k = 0$ $(k = 3, \dots, m)$, i.e. $x = 0$. This solution does not satisfy the last equation in (4.13). This means that there are no extreme points belonging to the interior of K. The maximum is therefore attained on the boundary of K, where at least one of the x_j's is zero. This shows that if (4.12) is true for $m-1$, then it is true also for m.

We have proved (4.12) assuming $0 < \mu_1 < \ldots < \mu_m$; in case $\mu_i = \mu_j$ for some i, j, it is obvious how to obtain the result for m from the one for $m-1$.

Finally, let us show that

$$\frac{(\mu_i + \mu_j)^2}{4\,\mu_i\mu_j} \leqslant \frac{(\mu_1 + \mu_m)^2}{4\,\mu_1\mu_m} \tag{4.17}$$

for any $1 \leqslant i, j \leqslant m$. Set $\mu_j = \alpha_j \mu_m$ and suppose $i \leqslant j$. We have $0 < \alpha_1 \leqslant \ldots \leqslant \alpha_m = 1$. Inequality (4.17) is equivalent to

$$\alpha_1(\alpha_i + \alpha_j)^2 \leqslant \alpha_i\alpha_j(\alpha_1 + 1)^2$$

i.e.

$$\alpha_1\alpha_i(\alpha_i - \alpha_j) + (\alpha_1\alpha_j - \alpha_i)\alpha_j \leqslant 0$$

and this is true, because $\alpha_i \leqslant \alpha_j$ and $\alpha_1\alpha_j \leqslant \alpha_1 \leqslant \alpha_i$. □

Theorem 5 *Let $\mathscr{A}$ be a real matrix $\{a_{hk}\}$ with $h, k = 1, \ldots, m$. Let us suppose $\mathscr{A} = \mathscr{A}^t$ and $\mathscr{A} \geqslant 0$ (in the sense $\langle \mathscr{A}(x)\xi, \xi\rangle \geqslant 0$, for almost every $x \in (a, b)$ and for any $\xi \in \mathbb{R}^m$). The operator A is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_m(x))^2 \leqslant \mu_1(x)\mu_m(x)$$

almost everywhere, where $\mu_1(x)$ and $\mu_m(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}(x)$ respectively. In the particular case $m = 2$ this condition is equivalent to

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\operatorname{tr} \mathscr{A}(x))^2 \leqslant \det \mathscr{A}(x)$$

almost everywhere.

Proof. From Theorem 4 A is L^p-dissipative if and only if (4.4) holds for almost every $x \in (a, b)$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$. We claim that in the present case this condition is equivalent to

$$\langle \mathscr{A}(x)\xi, \xi\rangle - (1 - 2/p)^2 \langle \mathscr{A}(x)\omega, \omega\rangle (\langle \xi, \omega\rangle)^2 \geqslant 0 \tag{4.18}$$

for almost every $x \in (a, b)$ and for any $\xi, \omega \in \mathbb{R}^m$, $|\omega| = 1$. Indeed, it is obvious that if

$$\langle \mathscr{A}(x)\lambda, \lambda\rangle - (1 - 2/p)^2 \langle \mathscr{A}(x)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2 \geqslant 0$$

for almost every $x \in (a,b)$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, then (4.18) holds for almost every $x \in (a,b)$ and for any $\xi, \omega \in \mathbb{R}^m$, $|\omega| = 1$. Conversely, fix $x \in (a,b)$ and suppose that (4.18) holds for any $\xi, \omega \in \mathbb{R}^m$, $|\omega| = 1$. Let Q be an orthogonal matrix such that $\mathscr{A}(x) = Q^t D Q$, D being a diagonal matrix. If we denote by μ_j the eigenvalues of $\mathscr{A}(x)$, we have

$$\begin{aligned}
&\langle \mathscr{A}(x)\lambda, \lambda\rangle - (1-2/p)^2 \langle \mathscr{A}(x)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2 \\
&= \langle DQ\lambda, Q\lambda\rangle - (1-2/p)^2 \langle DQ\omega, Q\omega\rangle (\mathscr{R}e\langle Q\lambda, Q\omega\rangle)^2 \\
&= \mu_j |(Q\lambda)_j|^2 - (1-2/p)^2 (\mu_j |(Q\omega)_j|^2)(\mathscr{R}e\langle Q\lambda, Q\omega\rangle)^2 \\
&\geqslant \mu_j |(Q\lambda)_j|^2 - (1-2/p)^2 (\mu_j |(Q\omega)_j|^2)(|(Q\lambda)_k|\,|(Q\omega)_k|)^2.
\end{aligned}$$

The last expression is nonnegative because of (4.18) and the equivalence is proved.

Let us fix $x \in (a,b)$. We may write (4.18) as

$$(1-2/p)^2 (\mu_h \omega_h^2)(\xi_k \omega_k)^2 \leqslant \mu_j \xi_j^2 \tag{4.19}$$

for any $\xi, \omega \in \mathbb{R}^m$, $|\omega| = 1$. Let us fix $\omega \in \mathbb{R}^m$, $|\omega| = 1$; inequality (4.19) is true if and only if

$$(1-2/p)^2 (\mu_h \omega_h^2) \sup_{\substack{\xi \in \mathbb{R}^n \\ \xi \neq 0}} \frac{(\xi_k \omega_k)^2}{\mu_j \xi_j^2} \leqslant 1.$$

We have

$$\max_{\substack{\xi \in \mathbb{R}^n \\ \xi \neq 0}} \frac{(\xi_k \omega_k)^2}{\mu_j \xi_j^2} = \mu_k^{-1} \omega_k^2;$$

in fact, by Cauchy's inequality, we have $(\xi_k \omega_k)^2 \leqslant (\mu_j \xi_j^2)(\mu_k^{-1} \omega_k^2)$ for any $\xi \in \mathbb{R}^m$ and there is equality if $\xi_j = \mu_j^{-1} \omega_j$.

Therefore (4.19) is satisfied if and only if

$$(1-2/p)^2 (\mu_h \omega_h^2)(\mu_k^{-1} \omega_k^2) \leqslant 1$$

for any $\omega \in \mathbb{R}^m$, $|\omega| = 1$, and (4.11) shows that this is true if and only if

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \frac{(\mu_1 + \mu_m)^2}{\mu_1 \mu_m} \leqslant 1\,.$$

The result for $m = 2$ follows from the identities

$$\mu_1(x)\mu_2(x) = \det \mathscr{A}(x), \quad \mu_1(x) + \mu_2(x) = \operatorname{tr} \mathscr{A}(x). \tag{4.20}$$

$\square$

Corollary 6 *Let $\mathscr{A}$ be a real and symmetric matrix. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k>0$ such that $A-kI(d^2/dx^2)$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[(1+\sqrt{p\,p'}/2)\,\mu_1(x)+(1-\sqrt{p\,p'}/2)\,\mu_m(x)\right]>0. \tag{4.21}$$

In the particular case $m=2$ conditions (4.21) *is equivalent to*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\operatorname{tr}\mathscr{A}(x)-\frac{\sqrt{p\,p'}}{2}\sqrt{(\operatorname{tr}\mathscr{A}(x))^2-4\det\mathscr{A}(x)}\right]>0. \tag{4.22}$$

Proof. *Necessity.* Corollary 3 shows that $\mathscr{A}(x)-kI\geqslant 0$ almost everywhere. In view of Theorem 5, we have that $A-kI(d^2/dx^2)$ is L^p-dissipative if and only if

$$\left(\frac{1}{p}-\frac{1}{2}\right)^2(\mu_1(x)+\mu_m(x)-2k)^2\leqslant(\mu_1(x)-k)\,(\mu_m(x)-k) \tag{4.23}$$

almost everywhere.

Inequality (4.23) is

$$\frac{4}{p\,p'}\,(\mu_1(x)+\mu_m(x)-2k)^2-(\mu_1(x)-\mu_m(x))^2\geqslant 0. \tag{4.24}$$

By Corollary 4, $A-k'I(d^2/dx^2)$ is L^p-dissipative for any $k'\leqslant k$. Therefore inequality (4.24) holds if we replace k by any $k'<k$. This implies that k is less than or equal to the smallest root of the left hand-side of (4.24), i.e.

$$k\leqslant\frac{1}{2}\left[(1+\sqrt{p\,p'}/2)\,\mu_1(x)+(1-\sqrt{p\,p'}/2)\,\mu_m(x)\right] \tag{4.25}$$

and (4.21) is proved.

Sufficiency. Let k be such that

$$0<k\leqslant\operatorname*{ess\,inf}_{x\in(a,b)}\frac{1}{2}\left[(1+\sqrt{p\,p'}/2)\,\mu_1(x)+(1-\sqrt{p\,p'}/2)\,\mu_m(x)\right]$$

Since $\mu_1(x)\leqslant\mu_m(x)$ and $\sqrt{p\,p'}/2\geqslant 1$, we have

$$(1+\sqrt{p\,p'}/2)\,\mu_1(x)+(1-\sqrt{p\,p'}/2)\,\mu_m(x)\leqslant 2\,\mu_1(x) \tag{4.26}$$

and then $\mathscr{A}(x) - kI \geqslant 0$ almost everywhere. The constant k satisifies (4.25) and this implies (4.24), i.e. (4.23). Theorem 5 gives the result.

The equivalence between (4.21) and (4.22) follows from the identities (4.20). $\square$

If we require something more about the matrix $\mathscr{A}$ we have also

Corollary 7 *Let $\mathscr{A}$ be a real and symmetric matrix. Suppose $\mathscr{A} \geqslant 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. If there exists $k > 0$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative, then*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\mu_1(x)\mu_m(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_m(x))^2\right] > 0. \tag{4.27}$$

If, in addition, there exists C such that

$$\langle \mathscr{A}(x)\xi, \xi\rangle \leqslant C|\xi|^2 \tag{4.28}$$

for almost every $x \in (a, b)$ and for any $\xi \in \mathbb{R}^m$, the converse is also true. In the particular case $m = 2$ condition (4.27) is equivalent to

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\det \mathscr{A}(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\operatorname{tr} \mathscr{A}(x))^2\right] > 0.$$

Proof. *Necessity.* By Corollary 6, (4.25) holds. On the other hand we have

$$\begin{aligned} &\left[(1 + \sqrt{p\,p'}/2)\,\mu_1(x) + (1 - \sqrt{p\,p'}/2)\,\mu_m(x)\right] \\ &\leqslant \left[(1 - \sqrt{p\,p'}/2)\,\mu_1(x) + (1 + \sqrt{p\,p'}/2)\,\mu_m(x)\right] \end{aligned}$$

and then

$$\begin{aligned} 4k^2 \leqslant &\left[(1 + \sqrt{p\,p'}/2)\,\mu_1(x) + (1 - \sqrt{p\,p'}/2)\,\mu_m(x)\right] \\ &\times \left[(1 - \sqrt{p\,p'}/2)\,\mu_1(x) + (1 + \sqrt{p\,p'}/2)\,\mu_m(x)\right]. \end{aligned}$$

This inequality can be written as

$$\frac{4k^2}{p\,p'} \leqslant \mu_1(x)\mu_m(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_m(x))^2$$

and (4.27) is proved.

Sufficiency. There exists $h > 0$ such that

$$h \leqslant \mu_1(x)\mu_m(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_m(x))^2$$

almost everywhere, i.e.

$$\begin{aligned} p\,p'h \leqslant & \left[(1 + \sqrt{p\,p'}/2)\,\mu_1(x) + (1 - \sqrt{p\,p'}/2)\,\mu_m(x)\right] \\ & \times \left[(1 - \sqrt{p\,p'}/2)\,\mu_1(x) + (1 + \sqrt{p\,p'}/2)\,\mu_m(x)\right] \end{aligned}$$

almost everywhere. Since $\mu_1(x) \geqslant 0$, we have also

$$(1 - \sqrt{p\,p'}/2)\,\mu_1(x) + (1 + \sqrt{p\,p'}/2)\,\mu_m(x) \leqslant (1 + \sqrt{p\,p'}/2)\,\mu_m(x) \quad (4.29)$$

and then

$$\begin{aligned} & (1 + \sqrt{p\,p'}/2)^{-1} p\,p'h \\ \leqslant & \left[(1 + \sqrt{p\,p'}/2)\,\mu_1(x) + (1 - \sqrt{p\,p'}/2)\,\mu_m(x)\right] \operatorname*{ess\,sup}_{y \in (a,b)} \mu_m(y) \end{aligned}$$

almost everywhere. By (4.28) $\operatorname{ess\,sup} \mu_m$ is finite and by (4.27) it is greater than zero. Then (4.21) holds and Corollary 6 gives the result. □

Remark 2 Generally speaking, assumption (4.28) cannot be omitted, even if $\mathscr{A} \geqslant 0$. Consider, e.g., $(a, b) = (1, \infty)$, $m = 2$, $\mathscr{A}(x) = \{a_{ij}(x)\}$ where $a_{11}(x) = (1-2/\sqrt{pp'})x + x^{-1}$, $a_{12}(x) = a_{21}(x) = 0$, $a_{22}(x) = (1+2/\sqrt{pp'})x + x^{-1}$. We have

$$\mu_1(x)\mu_2(x) - \left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1(x) + \mu_2(x))^2 = (8 + 4x^{-2})/(p\,p')$$

and (4.27) holds. But (4.21) is not satisfied, because

$$(1 + \sqrt{p\,p'}/2)\,\mu_1(x) + (1 - \sqrt{p\,p'}/2)\,\mu_2(x) = 2x^{-1}.$$

Corollary 8 *Let $\mathscr{A}$ be a real and symmetric matrix. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k>0$ such that $kI(d^2/dx^2)-A$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)}\Big[(1-\sqrt{p\,p'}/2)\,\mu_1(x)+(1+\sqrt{p\,p'}/2)\,\mu_m(x)\Big]<\infty. \tag{4.30}$$

In the particular case $m=2$ condition (4.30) is equivalent to

$$\operatorname*{ess\,sup}_{x\in(a,b)}\left[\operatorname{tr}\mathscr{A}(x)+\frac{\sqrt{p\,p'}}{2}\sqrt{(\operatorname{tr}\mathscr{A}(x))^2-4\det\mathscr{A}(x)}\right]<\infty.$$

Proof. The proof runs as in Corollary 6. We have that $kI(d^2/dx^2)-A$ is L^p-dissipative if and only if (4.23) holds, provided that

$$kI-\mathscr{A}(x)\geqslant 0$$

almost everywhere. Because of this inequality, we have to replace (4.25) and (4.26) by

$$k\geqslant\frac{1}{2}\Big[(1-\sqrt{p\,p'}/2)\,\mu_1(x)+(1+\sqrt{p\,p'}/2)\,\mu_m(x)\Big]$$

and

$$(1-\sqrt{p\,p'}/2)\,\mu_1(x)+(1+\sqrt{p\,p'}/2)\,\mu_m(x)\geqslant 2\,\mu_m(x) \tag{4.31}$$

respectively. □

In the case of a positive matrix $\mathscr{A}$, we have

Corollary 9 *Let $\mathscr{A}$ be a real and symmetric matrix. Suppose $\mathscr{A}\geqslant 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k>0$ such that $kI(d^2/dx^2)-A$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,sup}_{x\in(a,b)}\mu_m(x)<\infty. \tag{4.32}$$

Proof. The equivalence between (4.30) and (4.32) follows from (4.29) and (4.31). □

We have also

Corollary 10 *Let $\mathscr{A}$ be a real and symmetric matrix. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathscr{A}(x)$ respectively. There exists $k \in \mathbb{R}$ such that $A - kI(d^2/dx^2)$ is L^p-dissipative if and only if*

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[(1+\sqrt{p\,p'}/2)\,\mu_1(x) + (1-\sqrt{p\,p'}/2)\,\mu_m(x)\right] > -\infty.$$

In the particular case $m = 2$ this condition is equivalent to

$$\operatorname*{ess\,inf}_{x\in(a,b)} \left[\operatorname{tr}\mathscr{A}(x) - \frac{\sqrt{p\,p'}}{2}\sqrt{(\operatorname{tr}\mathscr{A}(x))^2 - 4\det\mathscr{A}(x)}\right] > -\infty.$$

Proof. The proof is similar to that of Corollary 6. □

4.3 L^p-dissipativity of the operator (4.1)

In this Section we consider the partial differential operator (4.1) with complex coefficients.

Here y_h denotes the $(n-1)$-dimensional vector $(x_1, \dots, x_{h-1}, x_{h+1}, \dots, x_n)$ and we set $\omega(y_h) = \{x_h \in \mathbb{R} \mid x \in \Omega\}$.

Lemma 5 *The operator (4.1) is L^p-dissipative if and only if the ordinary differential operators*

$$A(y_h)[u(x_h)] = d(\mathscr{A}^h(x)du/dx_h)/dx_h$$

are L^p-dissipative in $\omega(y_h)$ for almost every $y_h \in \mathbb{R}^{n-1}$ $(h = 1, \dots, n)$. This condition is void if $\omega(y_h) = \emptyset$.

Proof. *Sufficiency.* Suppose $p \geqslant 2$. If $u \in (C_0^1(\Omega))^m$ we may write

$$\mathscr{R}e \sum_{h=1}^{n} \int_\Omega \langle \mathscr{A}^h(x)\partial_h u, \partial_h(|u|^{p-2}u)\rangle dx =$$

$$\mathscr{R}e \sum_{h=1}^{n} \int_{\mathbb{R}^{n-1}} dy_h \int_{\omega(y_h)} \langle \mathscr{A}^h(x)\partial_h u, \partial_h(|u|^{p-2}u)\rangle dx_h.$$

By assumption

$$\mathscr{R}e \int_{\omega(y_h)} \langle \mathscr{A}^h(x)v'(x_h), (|v(x_h)|^{p-2}v(x_h))'\rangle dx_h \geqslant 0$$

for almost every $y_h \in \mathbb{R}^{n-1}$ and for any $v \in (C_0^1(\omega(y_h)))^m$, provided $\omega(y_h) \neq \emptyset$ $(h = 1, \ldots, n)$. This implies

$$\mathscr{R}e \sum_{h=1}^{n} \int_{\Omega} \langle \mathscr{A}^h(x) \partial_h u, \partial_h(|u|^{p-2}u) \rangle dx \geqslant 0.$$

The proof for $1 < p < 2$ runs in the same way. We have just to use (1.3) instead of (1.2).

Necessity. Assume first that $\mathscr{A}^h$ are constant matrices and $\Omega = \mathbb{R}^n$. Let $p \geqslant 2$ and fix $1 \leqslant k \leqslant n$.

Take $\alpha \in (C_0^1(\mathbb{R}))^m$ and $\beta \in C_0^1(\mathbb{R}^{n-1})$. Consider

$$u_\varepsilon(x) = \alpha(x_k/\varepsilon)\, \beta(y_k)$$

We have

$$\begin{aligned}
&\sum_{h=1}^{n} \int_{\mathbb{R}^n} \langle \mathscr{A}^h \, \partial_h u_\varepsilon, \partial_h(|u_\varepsilon|^{p-2} u_\varepsilon) \rangle dx = \\
&\varepsilon^{-2} \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^p dy_k \int_{\mathbb{R}} \langle \mathscr{A}^k \, \alpha'(x_k/\varepsilon), \gamma'(x_k/\varepsilon) \rangle \, dx_k \\
&+ \sum_{\substack{h=1 \\ h \neq k}}^{n} \int_{\mathbb{R}^{n-1}} \partial_h \beta(y_k) \, \partial_h(|\beta(y_k)|^{p-2} \beta(y_k)) \, dy_k \\
&\times \int_{\mathbb{R}} \langle \mathscr{A}^h \, \alpha(x_k/\varepsilon), \alpha(x_k/\varepsilon) \rangle \, |\alpha(x_k/\varepsilon)|^{p-2} dx_k \\
&= \varepsilon^{-1} \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^p dy_k \int_{\mathbb{R}} \langle \mathscr{A}^k \, \alpha'(t), (|\alpha(t)|^{p-2} \alpha(t))' \rangle \, dt \\
&+ \varepsilon \sum_{\substack{h=1 \\ h \neq k}}^{n} \int_{\mathbb{R}^{n-1}} \partial_h \beta(y_k) \, \partial_h(|\beta(y_k)|^{p-2} \beta(y_k)) \, dy_k \int_{\mathbb{R}} \langle \mathscr{A}^h \, \alpha(t), \alpha(t) \rangle \, |\alpha(t)|^{p-2} dt
\end{aligned}$$

where $\gamma(t) = |\alpha(t)|^{p-2}\alpha(t)$. Keeping in mind (1.2) and letting $\varepsilon \to 0^+$, we find

$$\mathscr{R}e \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^p dy_k \int_{\mathbb{R}} \langle \mathscr{A}^k \, \alpha'(t), (|\alpha(t)|^{p-2} \alpha(t))' \rangle \, dt \geqslant 0$$

and then

$$\mathscr{R}e \int_{\mathbb{R}} \langle \mathscr{A}^k \, \alpha'(t), (|\alpha(t)|^{p-2} \alpha(t))' \rangle \, dt \geqslant 0$$

for any $\alpha \in C_0^1(\mathbb{R})$. This shows that $A(y_k)$ is L^p-dissipative.

If $\mathscr{A}^h$ are not necessarily constant, consider

$$v(x) = \varepsilon^{(2-n)/2}\psi((x-x_0)/\varepsilon)$$

where $x_0 \in \Omega$, $\psi \in (C_0^1(\mathbb{R}^n))^m$ and ε is sufficiently small.

In view of Lemma 2 we write

$$\int_\Omega \Big(\mathscr{R}e\langle \mathscr{A}^h\, \partial_h v, \partial_h v\rangle - (1-2/p)^2 |v|^{-4}\, \mathscr{R}e\langle \mathscr{A}^h\, v, v\rangle (\mathscr{R}e\langle v, \partial_h v\rangle)^2$$
$$-(1-2/p)|v|^{-2}\, \mathscr{R}e(\langle \mathscr{A}^h\, v, \partial_h v\rangle - \langle \mathscr{A}^h\, \partial_h v, v\rangle)\, \mathscr{R}e\langle v, \partial_h v\rangle \Big) dx \geqslant 0$$

i.e.

$$\int_{\mathbb{R}^n} \Big(\mathscr{R}e\langle \mathscr{A}^h(x_0+\varepsilon z)\partial_h \psi, \partial_h \psi\rangle$$
$$-(1-2/p)^2|\psi|^{-4}\, \mathscr{R}e\langle \mathscr{A}^h(x_0+\varepsilon z)\psi, \psi\rangle (\mathscr{R}e\langle \psi, \partial_h\psi\rangle)^2$$
$$-(1-2/p)|\psi|^{-2}\, \mathscr{R}e(\langle \mathscr{A}^h(x_0+\varepsilon z)\psi, \partial_h \psi\rangle$$
$$-\langle \mathscr{A}^h(x_0+\varepsilon z)\partial_h\psi, \psi\rangle)\, \mathscr{R}e\langle \psi, \partial_h \psi\rangle \Big) dz \geqslant 0.$$

Letting $\varepsilon \to 0^+$, we obtain

$$\int_{\mathbb{R}^n} \Big(\mathscr{R}e\langle \mathscr{A}^h(x_0)\partial_h \psi, \partial_h \psi\rangle - (1-2/p)^2|\psi|^{-4}\, \mathscr{R}e\langle \mathscr{A}^h(x_0)\psi, \psi\rangle (\mathscr{R}e\langle \psi, \partial_h\psi\rangle)^2$$
$$-(1-2/p)|\psi|^{-2}\, \mathscr{R}e(\langle \mathscr{A}^h(x_0)\psi, \partial_h \psi\rangle - \langle \mathscr{A}^h(x_0)\partial_h\psi, \psi\rangle)\, \mathscr{R}e\langle \psi, \partial_h \psi\rangle \Big) dy \geqslant 0$$

for almost every $x_0 \in \Omega$.

Because of the arbitrariness of $\psi \in (C_0^1(\mathbb{R}^n))^m$, Lemma 2 shows that the constant coefficient operator $\partial_h(\mathscr{A}^h(x_0)\partial_h)$ is L^p-dissipative. From what has already been proved, the ordinary differential operators $(\mathscr{A}^h(x_0)v')'$ are L^p-dissipative $(h=1,\ldots,n)$.

Theorem 4 yelds

$$\begin{gathered} \mathscr{R}e\langle \mathscr{A}^h(x_0)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle \mathscr{A}^h(x_0)\omega, \omega\rangle (\mathscr{R}e\langle \lambda, \omega\rangle)^2 \\ -(1-2/p)\, \mathscr{R}e(\langle \mathscr{A}^h(x_0)\omega, \lambda\rangle - \langle \mathscr{A}^h(x_0)\lambda, \omega\rangle)\, \mathscr{R}e\langle \lambda, \omega\rangle \geqslant 0 \end{gathered} \tag{4.33}$$

for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega|=1$, $h=1,\ldots,n$.

Fix h and denote by N the set of $x_0 \in \Omega$ such that (4.33) does not hold for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega|=1$. Since N has zero measure, for almost every

$y_h \in \mathbb{R}^{n-1}$, the cross-sections $\{x_h \in \mathbb{R} \mid x \in N\}$ are measurable and have zero measure.

Hence, for almost every $y_h \in \mathbb{R}^{n-1}$, we have

$$\begin{gathered}\mathscr{R}e\langle \mathscr{A}^h(x)\lambda, \lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle \mathscr{A}^h(x)\omega, \omega\rangle (\mathscr{R}e\langle\lambda, \omega\rangle)^2 \\ -(1-2/p)\,\mathscr{R}e(\langle \mathscr{A}^h(x)\omega, \lambda\rangle - \langle \mathscr{A}^h(x)\lambda, \omega\rangle)\,\mathscr{R}e\langle\lambda, \omega\rangle \geqslant 0\end{gathered}$$

for almost every $x_h \in \omega(y_h)$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, provided $\omega(y_h) \neq \emptyset$. The conclusion follows from Theorem 4.

In the same manner we obtain the result for $1 < p < 2$. $\square$

Theorem 6 *The operator* (4.1) *is L^p-dissipative if and only if* (4.33) *holds for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \ldots, n$.*

Proof. *Necessity.* This has been already proved in the necessity part of the proof of Lemma 5.

Sufficiency. We have seen that if (4.33) holds for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, the ordinary differential operator $A(y_h)$ is L^p-dissipative for almost every $y_h \in \mathbb{R}^{n-1}$, provided $\omega(y_h) \neq \emptyset$ $(h = 1, \ldots, n)$. By Lemma 5, A is L^p-dissipative. $\square$

Remark 3 In the scalar case ($m = 1$), operator (4.1) falls into the operators considered in [4]. In fact, if $Au = \sum_{h=1}^n \partial_h(a^h \partial_h u)$, a^h being a scalar function, A can be written in the form (2.1) with $\mathscr{A} = \{c_{hk}\}$, $c_{hh} = a^h$, $c_{hk} = 0$ if $h \neq k$. The conditions obtained there can be directly compared with (4.33). The results of [4] show that operator A is L^p-dissipative if and only if (2.2) holds. This means that

$$\frac{4}{p\,p'}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta, \eta\rangle - 2(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi, \eta\rangle \geqslant 0 \tag{4.34}$$

almost everywhere and for any $\xi, \eta \in \mathbb{R}^n$ (see [4, Remark 1, p.1082]). In this particular case (4.34) is clearly equivalent to the following n conditions

$$\frac{4}{p\,p'}\,(\mathscr{R}e\,a^h)\,\xi^2 + (\mathscr{R}e\,a^h)\,\eta^2 - 2(1-2/p)(\mathscr{I}m\,a^h)\,\xi\eta \geqslant 0 \tag{4.35}$$

almost everywhere and for any $\xi, \eta \in \mathbb{R}$, $h = 1, \ldots, n$. On the other hand, in this case, (4.33) reads as

$$\begin{aligned} &(\mathscr{R}e\, a^h)|\lambda|^2 - (1-2/p)^2 (\mathscr{R}e\, a^h)(\mathscr{R}e(\lambda\overline{\omega})^2 \\ &-2(1-2/p)(\mathscr{I}m\, a^h)\, \mathscr{R}e(\lambda\overline{\omega})\, \mathscr{I}m(\lambda\overline{\omega}) \geqslant 0 \end{aligned} \tag{4.36}$$

almost everywhere and for any $\lambda, \omega \in \mathbb{C}$, $|\omega| = 1$, $h = 1, \ldots, n$. Setting $\xi + i\eta = \lambda\overline{\omega}$ and observing that $|\lambda|^2 = |\lambda\overline{\omega}|^2 = (\mathscr{R}e(\lambda\overline{\omega}))^2 + (\mathscr{I}m(\lambda\overline{\omega}))^2$, we see that conditions (4.35) (and then (4.34)) are equivalent to (4.36).

In the case of a real coefficient operator (4.1), we have also

Theorem 7 *Let A be the operator (4.1), where $\mathscr{A}^h$ are real matrices $\{a_{ij}^h\}$ with $i, j = 1, \ldots, m$. Let us suppose $\mathscr{A}^h = (\mathscr{A}^h)^t$ and $\mathscr{A}^h \geqslant 0$ ($h = 1, \ldots, n$). The operator A is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1^h(x) + \mu_m^h(x))^2 \leqslant \mu_1^h(x)\, \mu_m^h(x) \tag{4.37}$$

for almost every $x \in \Omega$, $h = 1, \ldots, n$, where $\mu_1^h(x)$ and $\mu_m^h(x)$ are the smallest and the largest eigenvalues of the matrix $\mathscr{A}^h(x)$ respectively. In the particular case $m = 2$ this condition is equivalent to

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\operatorname{tr} \mathscr{A}^h(x))^2 \leqslant \det \mathscr{A}^h(x)$$

for almost every $x \in \Omega$, $h = 1, \ldots, n$.

Proof. By Theorem 6, A is L^p-dissipative if and only if

$$\langle \mathscr{A}^h(x)\lambda, \lambda\rangle - (1-2/p)^2 \langle \mathscr{A}^h(x)\omega, \omega\rangle (\mathscr{R}e\langle\lambda, \omega\rangle)^2 \geqslant 0$$

for almost every $x \in \Omega$, for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \ldots, n$. The proof of Theorem 5 shows that these conditions are equivalent to (4.37). □

4.4 The angle of dissipativity

In this Section we find the precise angle of dissipativity for operator (4.1) with complex coefficients.

We first consider the ordinary differential operator (4.2) where $\mathscr{A}(x)$ is a matrix whose elements are complex locally integrable functions. Define the functions

$$\begin{gathered}
P(x,\lambda,\omega)=\mathscr{R}e\langle\mathscr{A}\,\lambda,\lambda\rangle-(1-2/p)^2\,\mathscr{R}e\langle\mathscr{A}\,\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2\\
-(1-2/p)\,\mathscr{R}e(\langle\mathscr{A}\,\omega,\lambda\rangle-\langle\mathscr{A}\,\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle;\\
Q(x,\lambda,\omega)=\mathscr{I}m\langle\mathscr{A}\,\lambda,\lambda\rangle-(1-2/p)^2\,\mathscr{I}m\langle\mathscr{A}\,\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2\\
-(1-2/p)\,\mathscr{I}m(\langle\mathscr{A}\,\omega,\lambda\rangle-\langle\mathscr{A}\,\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle
\end{gathered}\tag{4.38}$$

and denote by Ξ the set

$$\Xi=\{(x,\lambda,\omega)\in(a,b)\times\mathbb{C}^m\times\mathbb{C}^m\mid|\omega|=1,\ P^2(x,\lambda,\omega)+Q^2(x,\lambda,\omega)>0\}.$$

By adopting the conventions introduced in Lemma 1, we have

Theorem 8 *Let A be L^p-dissipative. The operator zA is L^p-dissipative if and only if*

$$\vartheta_-\leqslant\arg z\leqslant\vartheta_+$$

where

$$\vartheta_-=\operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi}(Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right)-\pi,$$
$$\vartheta_+=\operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi}(Q(x,\lambda,\omega)/P(x,\lambda,\omega))\right).$$

Proof. In view of Theorem 4 the operator $e^{i\vartheta}A$ is L^p-dissipative if and only if

$$\begin{gathered}
\mathscr{R}e\langle e^{i\vartheta}\,\mathscr{A}\,\lambda,\lambda\rangle-(1-2/p)^2\,\mathscr{R}e\langle e^{i\vartheta}\,\mathscr{A}\,\omega,\omega\rangle(\mathscr{R}e\langle\lambda,\omega\rangle)^2\\
-(1-2/p)\,\mathscr{R}e(\langle e^{i\vartheta}\,\mathscr{A}\,\omega,\lambda\rangle-\langle e^{i\vartheta}\,\mathscr{A}\,\lambda,\omega\rangle)\,\mathscr{R}e\langle\lambda,\omega\rangle\geqslant 0
\end{gathered}\tag{4.39}$$

for almost every $x\in(a,b)$ and for any $\lambda,\omega\in\mathbb{C}^m$, $|\omega|=1$.

By means of the functions $P(x,\lambda,\omega)$ and $Q(x,\lambda,\omega)$ introduced in (4.38), we can write (4.39) in the form

$$P(x,\lambda,\omega)\,\cos\vartheta-Q(x,\lambda,\omega)\,\sin\vartheta\geqslant 0.$$

Lemma 1 gives the result. □

Let now A be the partial differential operator (4.1). We have

Theorem 9 *Let A be L^p-dissipative. The operator zA is L^p-dissipative if and only if $\vartheta_- \leqslant \arg z \leqslant \vartheta_+$, where*

$$\vartheta_- = \max_{h=1,\dots,n} \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\right) - \pi,$$

$$\vartheta_+ = \min_{h=1,\dots,n} \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\right),$$

and

$$\begin{gathered} P_h(x,\lambda,\omega) = \mathscr{R}e\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle \mathscr{A}^h(x)\omega,\omega\rangle (\mathscr{R}e\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\, \mathscr{R}e(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\, \mathscr{R}e\langle\lambda,\omega\rangle, \end{gathered}$$

$$\begin{gathered} Q_h(x,\lambda,\omega) = \mathscr{I}m\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\, \mathscr{I}m\langle \mathscr{A}^h(x)\omega,\omega\rangle (\mathscr{R}e\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\, \mathscr{I}m(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\, \mathscr{R}e\langle\lambda,\omega\rangle, \end{gathered}$$

$$\begin{gathered} \Xi_h = \\ \{(x,\lambda,\omega)\in\Omega\times\mathbb{C}^m\times\mathbb{C}^m \mid |\omega|=1,\ P_h^2(x,\lambda,\omega)+Q_h^2(x,\lambda,\omega)>0\}. \end{gathered}$$

Proof. By Theorem 6, the operator $e^{i\vartheta}A$ is L^p-dissipative if and only if

$$\begin{gathered} \mathscr{R}e\langle e^{i\vartheta}\, \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\, \mathscr{R}e\langle e^{i\vartheta}\, \mathscr{A}^h(x)\omega,\omega\rangle (\mathscr{R}e\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\, \mathscr{R}e(\langle e^{i\vartheta}\, \mathscr{A}^h(x)\omega,\lambda\rangle - \langle e^{i\vartheta}\, \mathscr{A}^h(x)\lambda,\omega\rangle)\, \mathscr{R}e\langle\lambda,\omega\rangle \geqslant 0 \end{gathered} \tag{4.40}$$

for almost every $x\in\Omega$ and for any $\lambda,\omega\in\mathbb{C}^m$, $|\omega|=1$, $h=1,\dots,n$.

As in the proof of Theorems 8, conditions (4.40) mean $\vartheta_-^{(h)} \leqslant \vartheta \leqslant \vartheta_+^{(h)}$, where

$$\vartheta_-^{(h)} = \operatorname{arccot}\left(\operatorname*{ess\,inf}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\right) - \pi,$$

$$\vartheta_+^{(h)} = \operatorname{arccot}\left(\operatorname*{ess\,sup}_{(x,\lambda,\omega)\in\Xi_h} (Q_h(x,\lambda,\omega)/P_h(x,\lambda,\omega))\right),$$

and the result follows. $\square$

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